AIP Chaos
An Interdisciplinary Journal of Nonlinear Science

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Citation: Chaos: An Interdisciplinary Journal of Nonlinear Science **16**, 015103 (2006); doi: 10.1063/1.2165594
View online: http://dx.doi.org/10.1063/1.2165594
View Table of Contents: http://scitation.aip.org/content/aip/journal/chaos/16/1?ver=pdfcov
Published by the AIP Publishing

The size of the sync basin

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(Received 30 September 2005; accepted 19 December 2005; published online 31 March 2006)

We suggest a new line of research that we hope will appeal to the nonlinear dynamics community, especially the readers of this Focus Issue. Consider a network of identical oscillators. Suppose the synchronous state is locally stable but not globally stable; it competes with other attractors for the available phase space. How likely is the system to synchronize, starting from a random initial condition? And how does the probability of synchronization depend on the way the network is connected? On the one hand, such questions are inherently difficult because they require calculation of a global geometric quantity, the size of the "sync basin" (or, more formally, the measure of the basin of attraction for the synchronous state). On the other hand, these questions are wide open, important in many real-world settings, and approachable by numerical experiments on various combinations of dynamical systems and network topologies. To give a case study in this direction, we report results on the sync basin for a ring of $n \gg 1$ identical phase oscillators with sinusoidal coupling. Each oscillator interacts equally with its k nearest neighbors on either side. For k/n greater than a critical value (approximately 0.34, obtained analytically), we show that the sync basin is the whole phase space, except for a set of measure zero. As k/n passes below this critical value, coexisting attractors are born in a well-defined sequence. These take the form of uniformly twisted waves, each characterized by an integer winding number q, the number of complete phase twists in one circuit around the ring. The maximum stable twist is proportional to n/k; the constant of proportionality is also obtained analytically. For large values of n/k, corresponding to large rings or short-range coupling, many different twisted states compete for their share of phase space. Our simulations reveal that their basin sizes obey a tantalizingly simple statistical law: the probability that the final state has q twists follows a Gaussian distribution with respect to q. Furthermore, as n/k increases, the standard deviation of this distribution grows linearly with $\sqrt{n/k}$. We have been unable to explain either of these last two results by anything beyond a hand-waving argument.
 [DOI: 10.1063/1.2165594]

In the past few years, many researchers have become fascinated by a question that involves a fusion of nonlinear dynamics with network theory. The issue is to explore how the synchronizability of a network of oscillators depends on the way those oscillators are interconnected. Much has been learned by studying this question from a local perspective, using linearization to examine how the stability spectrum for the synchronous state depends on the network topology. Here we propose an alternative approach that focuses on a more global property of phase space: the basin of attraction for the synchronous state. The size of this basin controls the likelihood that a network will fall into sync. We suggest that there are many interesting discoveries to be made here, especially if one mixes and matches various networks and dynamical systems and conducts numerical experiments to explore how they affect synchronizability. To illustrate the sorts of questions one might ask, we present a case study of a ring of identical phase oscillators, and find that even here, intriguing patterns and puzzles pop up as soon as one begins to look for them.

I. INTRODUCTION

Many of the most interesting problems in science today involve large collections of dynamical systems connected together in complex networks.[1–3] From molecular biology to neuroscience, from condensed-matter physics to the Internet, researchers are unravelling the structure of complex networks, learning how they evolve and function, and exploring how their architecture affects the collective behavior they can display. It is in this last area that nonlinear science has so much to contribute.[4–22]

To take the simplest case, consider a network of dynamical systems that are identical, or nearly so. Under what conditions will such a network fall into sync, with all its elements acting as one? How does a network's ability to self-synchronize depend on its wiring diagram? And what is the best topology for achieving synchronization—or for avoiding it when it is undesirable?

From an applied perspective, a better understanding of how connectivity influences synchronization could yield benefits in several fields. For example,

- In computer science, Korniss *et al.*[20] have recently suggested a way to build a faster, more efficient architecture for massively parallel discrete-event simulations. Their analysis shows that the inclusion of a few random, long-range communication links between processors, in addition to the usual local connections, will help keep the distributed computation moving forward in step across the whole network, thereby avoiding the data-traffic bottlenecks that often plague such simulations.
- In applied physics, a similar use of sparse random connections is predicted to foster the coherent operation of superconducting Josephson ladder arrays.[21] On the other hand, such long-range links were far less effective at synchronizing two-dimensional arrays.
- In brain science, the different wiring patterns of two areas in the hippocampus seem to determine which form of epileptic activity they are more liable to exhibit.[22] Specifically, the smaller number of recurrent synaptic connections in the region known as CA1 makes it more prone to seizures, whereas the region known as CA3, with its greater recurrent connectivity, is more apt to show synchronous bursts instead.

The moral in each of these cases is that a network's architecture can strongly affect its propensity to synchronize.

The challenge now is to figure out the mathematical mechanisms at work here. One natural way to gain insight is to look at linearly coupled systems of identical oscillators, and then ask how the local stability of the synchronous state depends on the structure of the coupling matrix.[4–12] As pioneered by Pecora and co-workers,[4,5] this approach reveals that the synchronous state may or may not be stable, depending on where the eigenvalues of the coupling matrix (a topological concept) lie in relation to the roots of the "master stability function" (a dynamical concept). In this sense, synchronizability is indeed tied to network topology, as expected.

But because this framework is based on a local analysis, it is unable to shed light on a key global question: How *likely* is an oscillator network to synchronize, given that it has a stable synchronous state? This is a question about the basin of attraction for the synchronous state, which (giving in to temptation) we will refer to as the "sync basin."

The point is that linear stability analysis tells us nothing about how large or small this basin might be. Synchrony could be very stable and yet very unlikely, much as a golf ball is very stable once it reaches the bottom of the 18th hole on a golf course, but it is very unlikely to get there by following a random trajectory.

Because of its global character, the question of the likelihood of synchrony is far less tractable—and far less explored—than the local stability problem. Indeed, the entire topic of basins is something of an enigma in dynamical systems theory. We do know some numerical methods for approximating their boundaries, even in high-dimensional systems.[23] We know that these boundaries can be smooth or fractal, and we know that the basins themselves can be riddled with points from the basins of other attractors.[24] But what we do not know is how to compute the total volume or "measure" of a basin, which is what determines the probability that a random initial state will be drawn toward the associated attractor. Nor, for the problem of interest to us here, do we have any idea how the sync basin might expand or contract as the network topology is varied.

We first started thinking about the sync basin in the summer of 1999. At the time we were very interested in the ring model of small-world networks,[25] and we wondered whether the likelihood of synchrony might increase, perhaps dramatically, as the ring was progressively rewired from a lattice to a small world. The thought was that by randomly changing some local connections to long-range ones, the system might act in a much more coordinated fashion, thanks to the newly created communication channels spanning the whole network. The synchronizing effect of sparse, long-range connections was already suggested by earlier work in neuroscience,[26,27] so our guess seemed intuitively plausible.

For a warm-up problem, we looked at a one-dimensional ring of n identical phase oscillators, each coupled with equal strength to its k nearest neighbors on either side:

$$\dot{\phi}_i = \omega + \sum_{j=i-k}^{i+k} \sin(\phi_j - \phi_i), \quad i = 1, \ldots, n, \tag{1}$$

where $n \gg 1$ and the index i is periodic mod n. This system always has a stable synchronous state, given by $\phi_i = \omega t$ for all $i = 1, \ldots, n$. However, it was known that other attracting states, in the form of uniformly twisted traveling waves, were also possible under certain conditions.[28] So our intention was to rewire this system, by independently changing each edge to a random one with probability p or leaving it untouched with probability $1-p$, and then to study the likelihood of reaching the synchronous state as a function of p. But we quickly realized that we did not even understand the basin structure for $p=0$, the perfectly regular ring!

II. NUMERICAL EXPERIMENTS

A natural first question is to ask how the size of the sync basin for (1) depends on the coupling range k. Figure 1 shows that for small values of k, only a small fraction of initial conditions lead to synchrony. With increasing k, the sync basin expands to fill more and more of the phase space until eventually sync becomes the only attractor.

By zooming in on the transition more closely, we found that the synchronous state becomes globally stable when $k > k_c(n) \approx 0.34n$, or in other words, when each oscillator is coupled to about 68% of the others in the ring. In Sec. III we derive this result analytically and show that the prefactor 0.34 is given as the root of a certain transcendental equation. An exact result is possible here, because the critical value k_c can be obtained by a local analysis, even though the question (which deals with basins) is global. The trick is that the governing equations reduce to a gradient system in a suitable rotating frame, so all attractors are necessarily fixed points

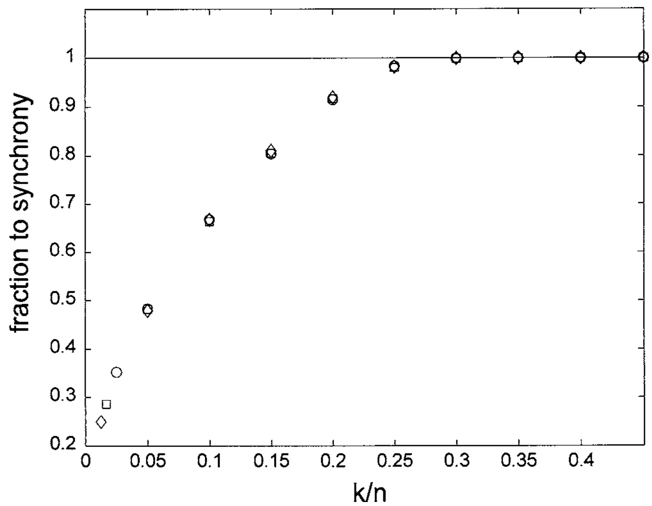

FIG. 1. Fraction of initial conditions that lead to perfect in-phase synchrony. Equation (1) was integrated numerically, starting from 100 000 uniformly random initial conditions, for various values of k and n. The circles, diamonds, and squares represent systems of size $n=80,60$, and 40, respectively. Error bars are smaller than the markers themselves. Note that the data collapse onto a single curve when plotted with respect to the dimensionless group k/n, which measures the coupling range as a fraction of the system size. Although the plot seems to suggest that 100% of initial conditions reach the synchronous state for $k/n=0.30$, in actuality the percentage measured is 99.99%. We observe the full 100% result for the data points with $k/n=0.35$.

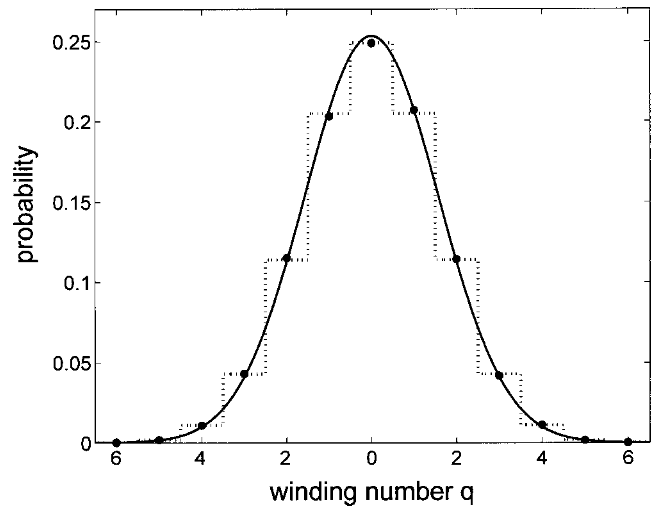

FIG. 2. Distribution of basin sizes for the various twisted states. These states are characterized by their winding number q. The data points represent results obtained from numerical integration of (1) with parameters $n=80,k=1$, starting from 100 000 uniformly random initial conditions. Because the winding number can only take integer values, we fit the data to a one-parameter discretized Gaussian where the probability that the final state has winding number q is defined by $(\sqrt{2\pi}\sigma)^{-1}\int_{q-1/2}^{q+1/2}\exp[-x^2/(2\sigma^2)]dx$. The discrete distribution, shown as a dotted histogram, represents the best (least squares) fit to the data and has a standard deviation of $\sigma=1.63\pm0.01$. The continuous curve reflects the corresponding continuous Gaussian distribution for that value of σ.

(corresponding to phase-locked periodic solutions in the original frame). When all of these, other than sync, are linearly unstable, we know by default that sync must be globally attracting.

As k decreases below k_c, other competing attractors are created. These take the form of uniformly twisted waves:

$$\phi_j = \omega t + 2\pi qj/n + C,$$

for $j=1,\ldots,n$. Here C is any constant and q is a winding number, an integer that measures the number of full twists in phase as we go around the ring once. The simulations indicate that the most gently twisted states, those with $q=\pm1$, are the first to become stable as k decreases. Then, with further reduction in k, more highly twisted states are stabilized in turn. (These results follow from a linear stability analysis about the q-twisted state, as shown in Sec. III.)

This raises the question of the relative sizes of the basins for the stable q-twisted states, compared with the sync basin. Figure 2 shows that the probability that the system will settle into a final state with q twists closely follows a normal distribution in q. Furthermore, as k decreases, the distribution broadens in a simple way: its standard deviation is well approximated by $\sigma\approx0.19\sqrt{n/k}-0.11$, as shown in Fig. 3.

We have no explanation for these statistical patterns, and we offer them as puzzles to the readers of this article.

The best anyone has come up with so far is the following heuristic argument, suggested by our colleague Jim Sethna. By assumption, the oscillators' phases are initially scrambled randomly from site to site, so at $t=0$ the concept of winding number is meaningless. But almost immediately, lots of violent phase slippage occurs; the phase pattern coarsens and smooths out as neighboring oscillators try to align with their neighbors. Since each oscillator interacts with k others on either side, a characteristic coherence length for the system at this stage should be roughly of size k. So perhaps the entire ring can now be viewed as a collection of n/k domains, each with a reasonably well-defined number of twists in its phase pattern. Assuming that the total number of phase twists in the solution is conserved from now on, one can estimate the

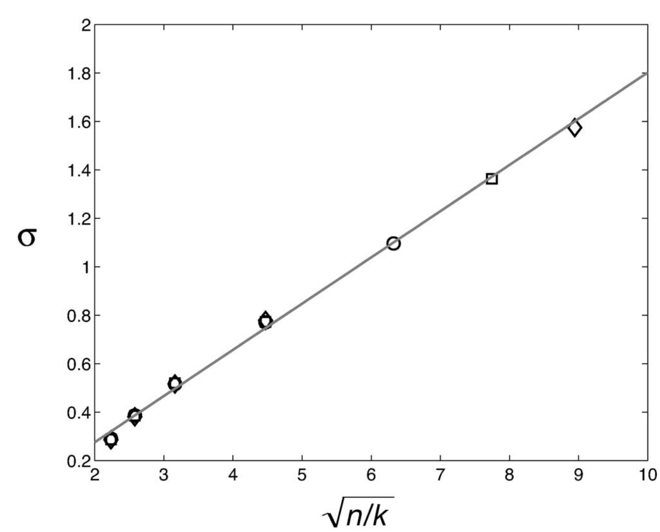

FIG. 3. Standard deviation of the distribution of basin sizes for different values of n/k. The circles, diamonds, and squares represent systems of size $n=80,60$, and 40, respectively. Error bars on the points are smaller than the markers themselves. The standard deviation σ for each condition was calculated using the procedure described in the caption of Fig. 2. As before, 100 000 random initial conditions were used to generate each data point. The solid line represents a least-squares fit of the data to $\sigma=a\sqrt{n/k}+b$, with $a=0.191\pm0.007$ and $b=-0.11\pm0.03$.

winding number by summing the twists contributed by each of the n/k domains. By the central limit theorem, this total twist should be normally distributed with a standard deviation that grows like the square root of the number of domains, that is, like $\sqrt{n/k}$. Obviously this argument leaves a lot to be desired (for instance, aside from its lack of rigor, it does not account for the coefficient 0.19). We hope that someone will come up with something better.

Finally, in Sec. IV we outline a research program for exploring the sync basin and its dependence on network topology in a much more general setting. We also elaborate on why we feel these questions may be fruitful and potentially important.

III. ANALYTICAL RESULTS

From now on we are going to view (1) in a rotating frame, so that phase-locked periodic solutions reduce to fixed points. Thus, let $\theta_i=\phi_i-\omega t$. Then θ_i satisfies

$$\dot{\theta}_i=\sum_{j=i-k}^{i+k}\sin(\theta_j-\theta_i),\quad i=1,\ldots,n. \tag{2}$$

Generalizing slightly, suppose the coupling strength between oscillators i and j is not necessarily either 0 or 1, but is given by some weight G_{ij}. To retain the rotational and reflectional symmetry of (2), suppose that the weights depend only on the separation $|i-j|$ between the oscillators. Then the system becomes

$$\dot{\theta}_i=\sum_{j=i-n/2}^{i+n/2}G_{i-j}\sin(\theta_j-\theta_i),\quad i=1,\ldots,n. \tag{3}$$

Phrased in terms of the signed separation $s=i-j$, we assume that the weights G_s, for $s=-n/2,\ldots,n/2$, are non-negative, symmetric about $s=0$, and decreasing as s increases out to the maximum (diametrically opposite) separation of $\pm n/2$.

Equation (3) is a gradient system. To see this, let θ denote the vector $(\theta_1,\ldots,\theta_n)$. Then one can check that (3) is equivalent to $\dot{\theta}=-\nabla V$, where the potential function is

$$V=-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}G_{i-j}\cos(\theta_j-\theta_i). \tag{4}$$

Thus all the trajectories of (3) flow monotonically downhill on this potential surface and asymptotically approach fixed points. In particular, we need not concern ourselves with the possibility of more complicated long-term behavior, such as limit cycles, attracting tori, or strange attractors for (3). Note that the fixed points could be either local minima of V (in which case they are stable) or saddles or local maxima (in which case they are unstable).

The function V also has a nice physical interpretation: it gives the potential energy of an XY spin system at zero temperature, with spin state θ and interaction strengths and connection topology dictated by G. Then the dynamics (3) imply that the spins reorient themselves so as to steadily lower the energy of the system.

Although the remainder of the analysis could be conducted on the discrete system (3), it is simpler and clearer to work with its continuum limit. The conclusions in either case are essentially the same, once n becomes moderately large. So from now on, consider the spatially continuous version of (3), given by

$$\frac{\partial\theta}{\partial t}=\int_{-\pi}^{\pi}G(x-y)\sin[\theta(y,t)-\theta(x,t)]dy, \tag{5}$$

where $\theta(x,t)$ is the phase of oscillator x at time t. The index variable x runs from $-\pi$ to π with periodic boundary conditions. As above, the kernel G provides nonlocal coupling between the oscillators. It is symmetric, non-negative, and decreases with the separation $|x-y|$ along the ring. For convenience $G(x)$ is normalized to have unit integral. Our coupling function $-\sin(x)$ is attractive, in the sense that it tends to pull neighboring oscillators into phase with one another.

A. Twisted states

It is straightforward to verify that

$$\theta(x,t)=qx$$

solves the continuum system (5) for any integer q. (This relies on the evenness of G and the oddness of the sine coupling function.) We will refer to this particular solution as the "q-twisted state."

We now analyze its linear stability. Let

$$\theta(x,t)=qx+\eta(x,t), \tag{6}$$

where $\eta\ll 1$. Keeping only linear terms in η produces the following equation:

$$\frac{\partial\eta}{\partial t}=\int_{-\pi}^{\pi}G(x-y)\cos[q(y-x)][\eta(y,t)-\eta(x,t)]dy. \tag{7}$$

Splitting the right-hand side into two terms yields

$$\frac{\partial\eta}{\partial t}=\int_{-\pi}^{\pi}G(x-y)\cos[q(x-y)]\eta(y,t)dy - \eta(x,t)\int_{-\pi}^{\pi}G(x-y)\cos[q(x-y)]dy. \tag{8}$$

To simplify this equation, we define a new function $H(x,q)=G(x)\cos(qx)$. Then (8) reduces to

$$\frac{\partial\eta}{\partial t}=H*\eta-\hat{G}(q)\eta, \tag{9}$$

where $*$ denotes convolution and $\hat{G}$ denotes the Fourier transform of G, defined as $\hat{G}(q)=\int_{-\pi}^{\pi}G(y)e^{iqy}dy$.

Next we calculate the eigenvalues of (9). Writing the eigensolutions in the form $e^{\lambda t}e^{imx}$, one can check that the eigenvalues are real and are given by

$$\lambda(m,q)=\frac{\hat{G}(q+m)+\hat{G}(q-m)}{2}-\hat{G}(q). \tag{10}$$

Here $m=1,2,\ldots$ is the mode number of the perturbation and q is any integer. [We ignore the trivial case $m=0$, which corresponds to a perturbation in which the same uniform constant is added to all the phases. By rotational symmetry, such a perturbation amounts to a time translation along the q-twisted solution, and hence has $\lambda(0,q)=0$ for all q, consistent with what we find from (10).]

Thus we see that the q-twisted state $\theta(x,t)=qx$ is linearly stable if and only if

$$\hat{G}(q+m)-2\hat{G}(q)+\hat{G}(q-m)<0, \tag{11}$$

for all $m=1,2,\ldots$. In this sense, stability is determined by a countably infinite set of conditions, which is not surprising, since the twisted state must be stable to perturbations of every possible mode number m. Checking all these conditions could be difficult, in general, for an arbitrary kernel G. But fortunately, for the special G of interest to us here, it turns out that one of these conditions is stricter than all the others; if it is satisfied, all the others hold automatically.

B. A sufficient condition for stability

The form of (11) resembles a finite-difference approximation to a second derivative. Thus it seems clear that the stability of the q-twisted state and the convexity properties of $\hat{G}(k)$ must be strongly related.

Let us pursue this idea for the continuum analog of the system (1). In this special case, each oscillator is equally coupled to all its neighbors, out to a certain range. We write the associated kernel as

$$G(x)=\begin{cases}\dfrac{1}{2\pi r}, & -\pi r\leqslant x\leqslant \pi r\\ 0, & |x|>\pi r,\end{cases} \tag{12}$$

meaning that each oscillator is coupled to a fraction r of the ring with strength $1/(2\pi r)$ and does not interact with the remaining portion of the ring. The Fourier transform is

$$\hat{G}(q)=\frac{\sin(\pi qr)}{\pi qr}. \tag{13}$$

Now we introduce several new variables to ease the notation. If we let

$$f(z)=\frac{\sin(\pi z)}{\pi z},$$

then the stability conditions (11) become

$$\tfrac{1}{2}[f(qr+mr)+f(qr-mr)]<f(qr), \quad m=1,2,\ldots . \tag{14}$$

This can be cleaned up further by writing

$$Q=qr$$

and

$$M=mr.$$

Then (14) becomes

$$\tfrac{1}{2}[f(Q+M)+f(Q-M)]<f(Q), \quad M=r,2r,\ldots . \tag{15}$$

Finally, set

$$S_Q(M)=\tfrac{1}{2}[f(Q+M)+f(Q-M)]-f(Q). \tag{16}$$

The stability question can now be reformulated in the following way: given the values of the winding number q and the coupling range r, the q-twisted state is stable if and only if

$$S_Q(M)<0, \quad M=r,2r,\ldots, \tag{17}$$

where $Q=qr$. So the next step is to extract enough information about the function $S_Q(M)$ to determine under what conditions these inequalities hold.

It helps to regard $S_Q(M)$ as a function defined on the entire real line, even though we only need to evaluate it on a discrete set of M values. The idea is that if the inequality $S_Q(M)<0$ holds for *all* real $M\neq 0$, then it certainly holds for the discrete set of values $\{r,2r,3r,\ldots\}$, which is precisely what we need to ensure stability. In other words, this approach will quickly give us a sufficient condition for stability. (Obtaining a condition that is both necessary and sufficient is slightly trickier and will come next.)

To gain intuition about the behavior of the function $S_Q(M)$, we have plotted its graph in Fig. 4 for two values of Q. Observe that in both cases the graph has even symmetry and passes through the origin with zero slope, facts which follow immediately from (16). For sufficiently small Q, say $Q=0.64$ as in Fig. 4(a), $S_Q(M)$ is negative everywhere except at the origin. But when Q is increased to 0.70 [Fig. 4(b)], the graph develops two positive bumps bracketing the origin. Therefore, by continuity with respect to Q, the function $S_Q(M)$ must lose its negative definiteness at a value $Q=\mu$ somewhere in the interval $0.64<Q<0.7$.

In fact, as these pictures suggest, $S_Q(M)$ remains negative definite until its graph becomes concave up at the origin, which happens when $S_Q''(0)$ changes from negative to positive. From (16) we calculate that $S_Q''(0)=f''(Q)$. Furthermore, one can show that the second derivative of $f(Q)=\sin\pi Q/(\pi Q)$ is negative in the interval $(-\mu,\mu)$ where

$$\mu\approx 0.6626.$$

Here μ is the smallest positive root of $f''(Q)$ and can be obtained by solving

$$\tan(\pi\mu)=\frac{2\pi\mu}{2-(\pi\mu)^2}. \tag{18}$$

Now remembering that $Q=qr$, we obtain the desired sufficient condition for stability:

Theorem 1: *Given* $r>0$*, the q-twisted state of* (5) and (12) *is stable if* $|q|r<\mu$.

To see what this means for the original system (1) with n oscillators, each of which is coupled to its k nearest neighbors on either side, we note that $2k/n$ plays the same role as r; it expresses the fraction of the entire ring that a single oscillator feels. Thus, setting $r=2k/n$, the sufficient condition becomes $|q|k<\mu n/2$. For example, if the coupling is nearest neighbor ($k=1$), twisted states are guaranteed to be stable if they have no more than $(\mu/2)n\approx 0.33n$ twists. Actually, slightly more twists can be tolerated before stability is lost, as we will see next.

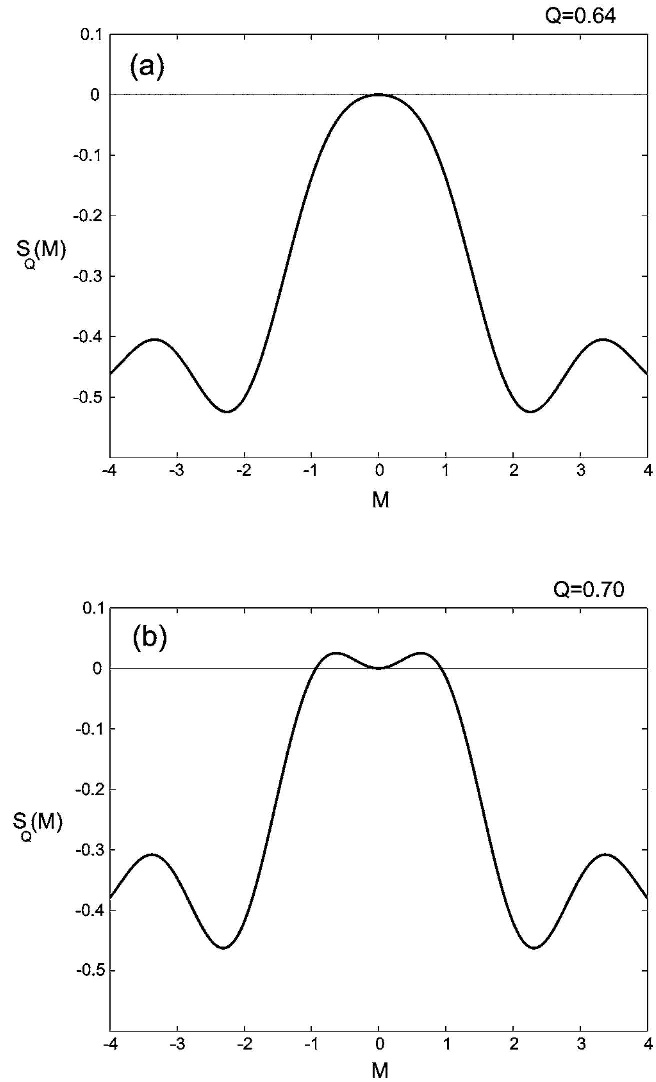

FIG. 4. Graphs of the function $S_Q(M)$ for two qualitatively different cases. (a) $Q=0.64$: The function $S_Q(M)$ is concave down at the origin and negative everywhere except at $M=0$, where $S_Q(0)=0$. (b) $Q=0.70$: The graph is now concave up at the origin, and there are two intervals on either side of $M=0$ where $S_Q(M)$ is positive.

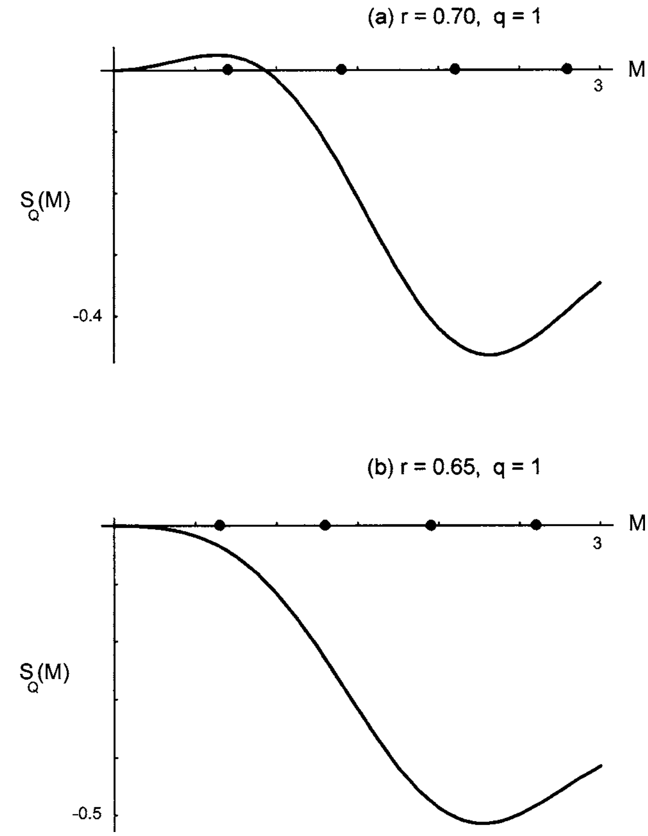

FIG. 5. Visualization of the necessary and sufficient condition for stability of the q-twisted state. The graph of $S_Q(M)$ (solid line) is shown in relation to the discrete set $\{M=1r,2r,3r,4r\}$ (black dots) for two different values of r. In both examples, $q=1$. (a) $r=0.70$: Here $Q=qr=0.70$. Since $Q>\mu\approx 0.6626$, the graph of $S_Q(M)$ is concave up at the origin and has a positive bump to the right of the origin. The leftmost black dot, at $M=1r$, lies in the interval where $S_Q(M)>0$; therefore the q-twisted state is unstable to perturbations along the associated mode $m=1$, as explained in the text. (b) $r=0.65$: Now $Q=qr=0.65<\mu$ and the graph is negative everywhere. This picture remains qualitatively unchanged for any $r\leq\mu$; all the black dots lie above the graph so the q-twisted state is stable. Thus the change in stability between (a) and (b) must occur for some intermediate value $r=r_c$ at which the graph of $S_Q(M)$ passes through the leftmost dot, creating a zero at $M=r_c$.

C. The critical first mode

To determine exactly where the q-twisted state changes stability, we need to examine the behavior of the function $S_Q(M)$ more carefully.

Suppose that $Q=qr$ is larger than μ, so that the graph of $S_Q(M)$ has two positive bumps on either side of the origin. Figure 5(a) shows such a case (for simplicity, only the right half of the bilaterally symmetric picture is shown). According to the stability criterion (17), the q-twisted state will be unstable if and only if any member of the discrete set $\{r,2r,\ldots\}$ lies inside the small interval where $S_Q(M)$ is positive. In the example shown in Fig. 5(a), only the leftmost point $M=r$ lies under the bump. This means that the twisted state would be unstable to perturbations along the first mode $m=1$; in other words, disturbances of the form $\eta=e^{\lambda t}e^{ix}$ would have $\lambda>0$ and hence grow exponentially.

Now consider what happens if we continuously decrease the coupling range r, holding q fixed. Then the picture in Fig. 5(a) will change in two ways simultaneously: $Q=qr$ will decrease, which pulls the bump in Fig. 5(a) downward and to the left; meanwhile, all the points in the discrete set $\{r,2r,\ldots\}$ slide to the left. The race is on—will the shrinking bump engulf the leftmost dot at $M=r$ before it can scurry away toward the origin? Yes, by continuity we know there *must* be a critical r where the graph of $S_Q(M)$ passes through the leftmost dot, since by the time Q falls below μ, the graph has already become negative definite so all the dots surely lie above it [Fig. 5(b)]. Hence the critical value r_c must correspond to a value of $Q_c=qr_c$ that is larger than μ.

This argument also shows that the first mode is always the critical one; when the twisted state is stable to infinitesimal perturbations along this mode, it is automatically stable to perturbations along any other mode.

By demanding stability with respect to the critical mode $m=1$, we can now boil the infinite number of conditions in (14) down to the following single, necessary, and sufficient condition for stability of the q-twisted state:

TABLE I. Computed values of r_c (the maximum value of the coupling range r for which a twisted state is stable) as a function of q, the number of full twists in the state. The value of $Q_c = qr_c$ rapidly approaches $\mu \approx 0.6626$ as q increases. The relative error is defined as $|(Q_c - \mu)/Q_c)|$.

q	$r_c(q)$	$Q_c = qr_c$	Relative error (%)
1	0.6809	0.6809	2.75
2	0.3333	0.6666	0.616
3	0.2214	0.6644	0.268
4	0.1659	0.6636	0.150
5	0.1326	0.6632	0.096

Theorem 2: *Given* $r > 0$, *the* q-*twisted state of* (5) and (12) *is stable if and only if*

$$\tfrac{1}{2}[f(qr + r) + f(qr - r)] < f(qr). \tag{19}$$

For a given value of q, this inequality becomes an equality precisely at $r = r_c$. Thus r_c can be obtained by numerically solving

$$f(qr_c + r_c) + f(qr_c - r_c) = 2f(qr_c).$$

Table I lists r_c as a function of q. Note that $qr_c > \mu \approx 0.6626$, as expected from the argument above. But the difference is not as large as one might have expected. Indeed, Table I shows that the product qr_c approaches μ very rapidly as q increases. The percentage difference between them is shown in Table I as the relative error $|(qr_c - \mu)/(qr_c)|$.

The most important number in Table I is $r_c = 0.6809$, the stability boundary for the $q = 1$ twisted state. This is the largest value of the coupling range r at which *any* twisted state can be stable. Once r exceeds that value, we strongly suspect that the only possible attractor is pure synchrony. Translating back to the finite-n system (1) by replacing r with $2k/n$, this would mean that for

$$k > k_c(n) \approx 0.34n,$$

sync is globally stable. Note that this theoretical prediction for $k_c(n)$ is consistent with the numerical results shown earlier in Fig. 1.

But in making this claim, we have glossed over one little thing. We are assuming that the only candidates for attractors are pure synchrony and the uniformly twisted states. We have not quite managed to prove this. Although we know from the earlier gradient system argument that all attractors must be fixed points, we have not yet ruled out the possible existence of stable fixed points that are *nonuniformly* twisted. It seems that some fixed points with spatially varying twist must exist—they are the only conceivable objects that could bifurcate from the uniformly twisted states when the latter lose stability—but so far we have not proven that all such states must be saddles. Nor have we ruled out more exotic fixed points, far from the uniformly twisted ones. We conjecture that any of these, if they exist, will be unstable.

IV. DISCUSSION

We hope that we have not exhausted your patience with the minutia of the previous section. The analysis there is a standard local calculation, and we did it in part because we could, and in part because it was the only way we could think of to shed any light on the global questions that really interest us. Specifically, the analysis goes a long way toward explaining the numerical observation in Fig. 1 that sync is globally stable for $k > k_c(n) \approx 0.34n$. But this sort of result is much weaker than what we want, which is a better understanding of the basin size distribution for the system (Fig. 2) and how it changes with network architecture (a role played in Fig. 3 by the ratio n/k).

Why does any of this matter for science more generally? There are a few reasons, both practical and theoretical. A better understanding of the sync basin, even a very crude and incomplete understanding, would be valuable in diverse fields. For instance, a healthy human heart has at least two competing attractors: the normal rhythm (analogous to sync, and hopefully with a huge basin of attraction) and ventricular fibrillation, a lethal arrhythmia that is stable once initiated and which accounts for hundreds of thousands of cases of sudden cardiac death every year in otherwise fit individuals. Likewise, the power grid, the largest machine ever built, is dynamically stable when functioning properly, but also when blacking out. Knowing more about the basin structure in both of these examples might help us develop heuristics for staying in the desirable region.

Admittedly these two problems are formidable and may not be the best place to start. So how about something as idealized as a Petri dish full of the Belousov-Zhabotinsky chemical reaction? The excitable version of this reaction (as opposed to the spontaneously oscillatory version) has a stable quiescent state of bland uniformity, colored rusty red everywhere, coexisting with patterned spatiotemporal states, adorned by one or more pairs of beautiful blue counter-rotating spiral waves. Anyone who has ever played around with this reaction knows that if you start it from a complicated initial condition (say, by sloshing the liquid in the dish to shear an existing pattern), the system is much more likely to settle into spiral waves than uniform quiescence. Why is that? And how does the probability of uniformity depend on the size of the dish?

The point is that hardly anyone is asking such questions, and we have hardly any techniques for answering them or even approaching them. This is a sign of opportunity. Any work on basins in complex networks or spatially extended systems is bound to uncover interesting things quickly.

There are so many natural questions to ask. Pick your favorite dynamical system, any type of network topology, any weighting scheme for the links, and ask how the probability of sync depends on those factors. Although we have no strategies for making analytical headway, good ideas might dawn on us after numerical experiments reveal the basic regularities at work here.

In particular, it should be numerically straightforward to check what rules (if any) govern the size of the sync basin. In place of n/k in Fig. 1 or the rewiring parameter p we originally intended to use, you could plot your results versus whatever parameter controls the network topology. For example, random graphs with prescribed degree distributions[1–3] can often be characterized by a single parameter, such as the mean of a Poisson degree distribution, the exponent of a power-law distribution, and so on.

With any luck, maybe the powerful methods of statistical mechanics can be brought to bear. This approach has already proven useful for calculating the storage capacity of associative-memory neural networks or exploring the basin structure of the Kauffman model of Boolean gene networks. The promising questions here would be to investigate how basin structure changes with network topology. Some work in this direction have already begun to appear.[29,30]

We wish you happy hunting.

ACKNOWLEDGMENTS

Research is supported in part by grants and fellowships from the Ford Foundation, the Alfred P. Sloan Foundation, and the National Science Foundation. These studies were conducted under the auspices of Cornell's Mathematical and Theoretical Biology Institute and IGERT Program in Nonlinear Systems. We thank Carlos Castillo-Chavez for encouragement and support, Danny Abrams for help in preparing the manuscript, and Jim Sethna for suggesting the heuristic argument for the scaling of the basin size distribution.

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