Supplemental Material

Section I presents additional analytical results for the Scrambler model. These include asymptotic results about the long-time behavior of the individual cluster densities c_i described in the main text, as well as an exact solution for the time-dependence of the *moments* of the c_i.

Section II modifies the Scrambler oscillator model to allow alternative couplings between oscillators. Specifically, we change how close to threshold an oscillator has to be in order to be absorbed by a firing cluster. This only modestly changes the analysis.

Section III examines a more substantial alteration of the model, in which we change its character from stochastic to deterministic. Now, when an oscillator reaches threshold, it no longer scrambles every other oscillator to a new, random, voltage. Instead, it kicks every other oscillator up by a constant amount, or up to threshold, whichever is less. This deterministic resetting rule is in line with the simplest traditional models of pulse-coupled oscillators. As will be shown, this change makes for a more involved analysis, but the results are qualitatively similar to those obtained in the main text. This qualitative agreement is what originally motivated our use of the Scrambler model in the first place.

Section IV explores how some of our results break down if we relax the assumption $N_j \gg 1$ made in the main text.

I. ANALYTICAL RESULTS

Asymptotic Behavior

In this subsection we investigate the long-time behavior of the cluster densities $c_i(t)$. For large t, we find that about 45% of all clusters in the system are singletons, while about 14% are doublets. Indeed, it seems likely that for any cluster size k, the fraction $c_k(t)/c(t)$ tends to a nonzero constant as $t \to \infty$. We were quite surprised by these results, since usually in aggregation the fraction of singletons decays to zero as the time grows (and the same holds true for all other species of clusters).

To derive these results, we begin by analyzing $c_1(t)$. Figure 1 in the main text indicates that $c_1(t)$ decays at a rate comparable to that of the disorder parameter $c(t)$. Thus it is natural to study their ratio $c_1(t)/c(t)$ for large t. Equations (1) and (4) of the main text showed that

$$c(t) = \exp(-t/2)$$

and

$$c_1(t) = \exp[-t + \mathrm{Ei}(-1) - \mathrm{Ei}(-e^{-t/2})],$$

where Ei denotes the exponential integral. If one expands $\mathrm{Ei}(-e^{-t/2})$ in the large-t limit, one finds that

$$\lim_{t\to\infty} \frac{c_1(t)}{c(t)} = \exp[\mathrm{Ei}(-1) - \gamma] = 0.45\ldots,$$

where γ is the Euler constant. Thus, asymptotically about 45% of all clusters are singletons.

To calculate the asymptotic fraction of doublets, we rewrite Eq.(6) of the main text as

$$\dot{c_2} = -c_2\left(1 - \frac{1}{2}\,e^{-2c}\right) + \frac{1}{2}\,c_1^2\,e^{-c}. \tag{1}$$

Next we solve this equation subject to $c_2(0) = 0$. The resulting expression for $c_2(t)$ is cumbersome, but it looks slightly simpler if we use c rather than t as the independent variable:

$$\frac{c_2}{c} = \exp\left(-\int_c^1 dx\,\frac{1-e^{-2x}}{x}\right)\mathcal{E}(c) \tag{2}$$

with

$$\mathcal{E}(c) \equiv \int_c^1 dy\,\exp\left[-y - \int_y^1 dx\,\frac{(1-e^{-x})^2}{x}\right].$$

Finally, since $c(t) \to 0$ as $t \to \infty$, we replace the c's in the lower limits of the integrals with $c = 0$ and thereby obtain (after numerical quadratures)

$$\lim_{t\to\infty} \frac{c_2(t)}{c(t)} = 0.14\ldots,$$

which shows that asymptotically, about 14% of all clusters are doublets.

Figure 1 shows that these predictions agree reasonably well with simulations.

Moments

In the main text, we derived the following rate equation for the individual cluster densities c_i:

$$\dot{c_i} = -c_i + \sum_{k=1}^{i} \frac{c_k}{2} e^{-kc} \sum_{\sum p a_p = i-k} \left(\prod_{p\geq 1} \frac{(kc_p)^{a_p}}{a_p!}\right). \tag{3}$$

Only c_1 and c_2 had closed-form solutions, so we resorted to numerical integration for the higher c_i. While we cannot analytically solve for all these higher c_i, we can solve for their moments, defined as

$$M_n(t) = \sum_j j^n c_j(t).$$

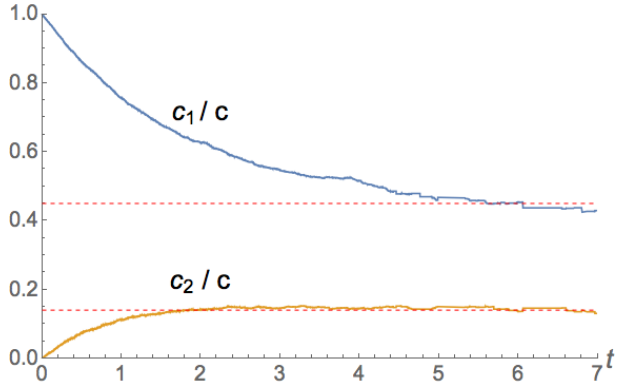

FIG. 1: Relative fraction of singletons and doublets, $c_1(t)/c(t)$ and $c_2(t)/c(t)$, for $N = 10^4$ oscillators. Red dotted lines show predicted asymptotic values, derived above.

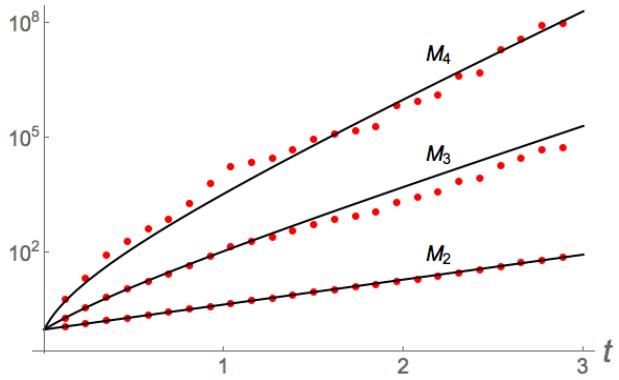

FIG. 2: Log plot of the first three nontrivial moments M_2, M_3, M_4. Black curves, theoretical predictions obtained from (5); red dots, average simulation results for 100 realizations of $N = 10^4$ oscillators.

The first two moments are trivial: $M_0 = \sum_j c_j(t) = c(t)$ and $M_1(t) = \sum_j j c_j(t) = 1$, from conservation of oscillators. The higher moments can be obtained from a generating function. Let $G(z,t) = \sum_{k\geq 1} c_k(t) e^{kz}$. Then the infinite set of differential equations (3) transforms into

$$\frac{\partial G(z,t)}{\partial t} + G(z,t) = \frac{1}{2} G[z - c(t) + G(z,t), t]. \qquad (4)$$

This equation looks neat, but it is far from trivial, as the right-hand side involves G in a very nonlinear manner. Using the identity $M_n(t) = \frac{\partial^n G}{\partial z^n}|_{z=0}$, we can however derive the following equations for the moments:

$$\begin{aligned} \dot{M}_2 &= \frac{3}{2} M_2 \\ \dot{M}_3 &= \frac{7}{2} M_3 + 3(M_2)^2 \\ \dot{M}_4 &= \frac{15}{4} M_4 + 16 M_2 M_3 + \frac{3}{2}(M_2)^3. \end{aligned} \qquad (5)$$

Like the c_i equations (Eq. (6) in the main text), these moment equations are recursive and can be solved in succession, except that here it is possible to do so explicitly. We find

$$\begin{aligned} M_2(t) &= e^{3t/2} \\ M_3(t) &= 7e^{7t/2} - 6e^{3t} \\ M_4(t) &= \frac{448}{5} e^{5t} - 128 e^{9t/2} + \frac{217}{5} e^{15t/4} - 4e^{27t/8}. \end{aligned} \qquad (6)$$

Figure 2 plots theoretical and simulated values of the M_i. The agreement is good for M_2 but worse for M_3 and M_4. This is to be expected. Each $c_i(t)$ is a stochastic process, subject to fluctuations dominated by the chance formation of big clusters. Since $M_n(t) = \sum_j j^n c_j(t)$, the higher moments amplify these fluctuations more and are therefore noisier themselves.

II. ALTERNATIVE COUPLINGS

We first restate the dynamics of the original Scrambler model, and then describe the variations.

Recall that in the main text we considered a population of $N \gg 1$ integrate-and-fire oscillators coupled all-to-all. Each oscillator was characterized by a voltage-like state variable x, which increased linearly according to $\dot{x}_i = 1$. When a cluster of j oscillators reached a threshold value set to 1, they fired and then instantly did three things: (i) they reassigned every other oscillator (or cluster of oscillators) a new voltage uniformly at random (they scrambled the oscillators) (ii) they absorbed any oscillators within a distance j/N of threshold and (iii) they reset their voltage to 0 along with any oscillators they absorbed.

We now modify event (ii) in either of two ways: when a cluster of size j fires, it either absorbs all oscillators within a new distance of threshold given by (a) kj/N or (b) k/N. Modification (a) generalizes the original model by including an adjustable coupling k. Modification (b) assumes that the absorption region is independent of j, the size of the firing cluster.

These generalizations change the analysis of the Scrambler model only slightly. For instance, to find the disorder parameter $c(t)$, we again use the rate equation $\dot{c} = -\sum_i R_i L_i$, where R_i denotes the rate at which a cluster of size i fires, and L_i denotes the number of oscillators absorbed when a cluster of size i fires. In the original model, to find L_i and R_i, we made liberal use of the fact that all oscillators on the interval $[1 - j/N, 1)$ were captured when a cluster of size j fired. With the generalized couplings (absorption distances), this interval simply changes to $[1 - kj/N, 1)$ and $[1 - k/N, 1)$. This change propagates through the analysis straightforwardly. Hence, we state the results in the following table without derivation. (In the table, $W(x)$ refers to Lambert's W function.)

	j/N	kj/N	k/N
L_i	ic	kic	kc
R_i	$c_i/2$	$c_i/(1+k)$	$c_i/(1+kc)$
$c(t)$	$e^{-t/2}$	$e^{-\frac{k}{k+1}t}$	$1/(kW(k^{-1}e^{(k^{-1}+t)}))$

The rows of the table give the results for L_i, R_i and $c(t)$; the columns show how the results vary for the three coupling schemes: original, (a), and (b). For coupling scheme (a), where the absorption distance is kj/N, the exponential decay constant in $c(t)$ is predicted to be $-k/(k+1)$. To test this, we simulated $c(t)$ for various k, and found the exponents of best fit. Figure 3 shows the results along with the theoretical curve.

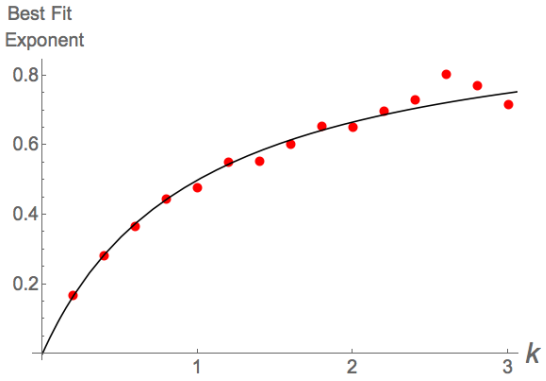

FIG. 3: Magnitude of the decay constant in $c(t)$ for coupling scheme (a). Black curve, theoretical prediction $k/(k+1)$; red dots, simulation results for $N = 5000$ oscillators.

We find similarly good agreement between theory and simulations for coupling scheme (b), as shown in Figure 4. The chief difference in this case is that the disorder parameter $c(t)$ decays algebraically as opposed to exponentially. To see this, consider the Taylor expansion of $c(t)$ for large t and $k = 1$:

$$c(t) = 1/(W(1^{-1}e^{(1+t)})) \approx \frac{1}{t} + O\left(\frac{1}{t^2}\right) \qquad (7)$$

as $t \to \infty$.

Intuitively, the physical reason for the non-exponential decay is that model (b) assumes that large clusters fire with the same strength as small ones. In contrast, the original model displayed exponential growth of synchrony (or equivalently, exponential decay of the disorder parameter $c(t)$) because it assumed that clusters fire with strength proportional to their size, which sets up a positive feedback loop in which the big clusters get bigger at the expense of smaller ones (because they fire more strongly and therefore absorb other oscillators in a snowballing fashion). That is why the level of synchrony grows exponentially fast in the original model, but not in the modified model.

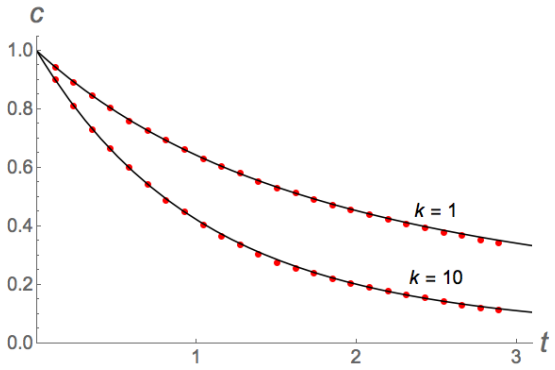

FIG. 4: Decay of the disorder parameter $c(t)$ for coupling scheme (b), which assumes a fixed absorption distance k/N. Black curves, theoretical predictions; red dots, simulation results for $N = 5000$ oscillators. Plots for $k = 1$ and $k = 10$ are shown.

Having solved for $c(t)$ in models (a) and (b), we could now go on to solve for the individual cluster densities c_i and the moments M_i. Nothing qualitatively new happens (compared to what we saw in the main text for the original model), so we omit the details.

III. DETERMINISTIC OSCILLATORS

The Scrambler model is a toy model. We introduced it to give the simplest possible mean-field model of pulse-coupled oscillators. Specifically, it was the random shuffling of oscillators during each firing event that simplified their analysis. It conveniently kept the voltages of all oscillators (and all clusters of oscillators) uniformly distributed on $[0, 1]$ at all times.

The extreme randomness of this resetting rule, however, is contrived. Traditional models of pulse-coupled oscillators, such as those analyzed by Peskin [1] and Mirollo and Strogatz [2], obey deterministic resetting rules, much as real biological oscillators obey deterministic phase-response curves. As we will show in this section, models with deterministic resetting can be reasonably approximated within the framework developed here.

For simplicity, we will restrict attention to an almost absurdly idealized model of pulse-coupled oscillators, even more idealized than the models discussed in Refs. [1, 2]. It consists of what we will refer to as Firefly oscillators. (Fictional Firefly oscillators would be a more apt description, given that essentially everything about the model is unrealistic for real fireflies.)

The equations of motion for the Fireflies are the same as for the Scramblers: $\dot{x}_i = 1$ (in between firing events). The initial voltages are again drawn from a uniform distribution. However, when a cluster of j synchronized Fireflies reaches the threshold value of 1, that cluster does just two things: (i) it imparts a voltage pulse of

size j/N to all other oscillators. Any subsequent oscillators that reach threshold by virtue of this extra j/N, and hence get absorbed by the firing cluster, do not fire until the *next* time they reach threshold. As before, this is to avoid complications that would be caused by chain reactions of firings. (ii) The cluster resets to $x = 0$ along with any oscillators it absorbed.

Firefly oscillators of this type can be viewed as a special case, in two respects, of the oscillators studied by Peskin [1] and Mirollo and Strogatz [2]. First, the oscillators of Refs. [1, 2] have a nonlinear charging curve rather than a linear one; their voltage dynamics are governed by $\dot{x_i} = S_0 - \gamma x_i$, where the parameters $S_0 > 0$ and $\gamma \geq 0$. Second, when an oscillator fires in the models of [1, 2], it imparts a voltage pulse of size ϵ to all other oscillators. The long-term behavior of the system then depends on the values of ϵ and γ.

Thus, the Firefly model studied here corresponds to the case $S_0 = 1$, $\gamma = 0$, and $\epsilon = 1/N$. For these parameters, the system almost always achieves global synchrony [3]. Our goal, then, is not to establish how the long-term behavior depends on parameters, since we already know that synchrony is inevitable for this simple model. Rather, the goal is to quantify how synchrony builds up over time. To put it another way, we want to predict the kinetics of cluster formation, growth, and coalescence as the system evolves toward synchrony.

Fireflies vs. Scramblers

Will the deterministic Fireflies behave (as hoped) like the stochastic Scramblers? There are two differences between them that complicate the analysis of the Fireflies.

The first difference has to do with how their speed $\dot{x}$ evolves as the system moves toward complete synchrony. The average speed of the Fireflies doesn't remain constant at unity, as it does for the Scramblers. This is because when a cluster of size j fires, all the other oscillators receive a pulse of size j/N which boosts them up on their voltage curve (whereas the Scramblers were just randomly reassigned on $[0, 1]$, and so keep the same speed $v_i = 1$ on average). The speeds of the Fireflies are thus $v_i = 1 + v_{\rm pulse}$, where $v_{\rm pulse}$ must be determined.

A second difference is that the firing rate of the Fireflies is piecewise constant (unlike that of the Scramblers, which as shown in the main text is a smooth function of time: $R_i(t) = c_i(t)/2$.) To see why the firing rate for the Fireflies is piecewise constant, first define Δx_i as the distance between the i^{th} and $(i+1)^{th}$ oscillators (or cluster of synchronized oscillators). Second, divide the temporal evolution into periods $\{T_n\}$, where each period is the time taken for the full population to complete a full cycle. Then ask, how does Δx_i behave during the first period, the time taken for the first wave of oscillators to complete their first cycle? Since all oscillators receive the same number of pulses and have the same speed, we see that each Δx_i won't change while the oscillators complete their virgin ascent through [0,1]. This implies a constant firing rate during this first period.

We will later show that this is not unique to the first period: the firing rates will take different, but constant, values during each period; they will be piecewise constant. This is in stark contrast to the Scramblers, where each Δx_i is constantly changing as the oscillators get reshuffled on $[0, 1]$ during each firing event.

Intuitive Picture of the Dynamics

With these differences in mind, we begin with a qualitative description of the dynamics. As mentioned, the speed of each oscillator, and the effect of a pulse on each oscillator, is the same. This means that the initial ordering of oscillators, or clusters of oscillators, will be invariant throughout the dynamics. They all march forward through $[0, 1]$ in a line with no passing.

Then we consider the *average* behavior as $N \to \infty$. In this limit, the average spacing between oscillators $\langle \Delta x_i \rangle$ approaches N^{-1}. Now, what happens when the first oscillator fires? It captures all oscillators on the interval $[1 - 1/N, 1)$. Since $\langle \Delta x_i \rangle = N^{-1}$, there will be exactly one oscillator on this interval, on average, and so one oscillator will be captured. This procedure will repeat itself for the next oscillator that fires, and the oscillator after that, such that every oscillator that fires captures the one behind itself. In this mean-field sense, then, the first wave of oscillators will be an orderly sequence of fire/capture/fire/capture, so that at the end of the first period, all oscillators will have synchronized into pairs spaced equally apart.

Of course, Δx_i will have fluctuations about the mean value of N^{-1}. For the oscillators spaced such that $\Delta x_i < N^{-1}$, no captures will take place. For $\Delta x_i > N^{-1}$, at least one capture will take place, and possibly more. So, at $t = T_1$ there will be a number of clusters of different sizes. It is not clear how these clusters are distributed on $[0, 1]$. Say, for example, that mostly clusters of size 2 and 3 formed, while a cluster of size 4 was the first capture, and a cluster of size 6 was the last capture. Then the clusters of sizes 2 and 3 will be approximately uniformly distributed in voltage, while the distribution of those of sizes 4 and 6 will be more sharply peaked.

Nevertheless we assume that the clusters of size i are uniformly distributed on $[0, 1]$ for each i. We recognize that this won't be accurate for each i for all values of t. It will however be accurate for those values of i which contain most of the oscillator "mass" and less so for those with less of the mass. So, our assumption will be imprecise for those c_i which are small, but since they are small, the inaccuracy won't matter, to first order.

Now that we understand the first period, how will the second period proceed? We again consider the *average* behavior as $N \to \infty$. In this mean-field description, we earlier concluded that at the end of the first period, all oscillators would have synchronized into pairs spaced equally apart on $[0,1]$. When the first pair fires, therefore, there will again be exactly one pair of synchronized oscillators behind them, and so as before, the second period will be an orderly sequence of fire/capture/fire/capture, resulting in all oscillators having synchronized into clusters of size 4.

Continuing this logic, we see the size of clusters will double during each period. Moreover, we observed that the clusters of oscillators will begin each period evenly spaced from each other. This implies the aforementioned piecewise constant firing rate. The mean-field dynamics are therefore trivial: there is a train-like progression of clusters of the same size through $[0,1]$, with each cluster that reaches threshold doubling in size by absorbing the cluster behind it.

Mean-field Analysis

With this picture in mind, we analyze the rate equation for our disorder parameter: $\dot{c} = -\sum_i R_i L_i$.

To calculate L_i, we again find all oscillators in the interval $[1-i/N, 1)$. It is tempting to write down $L_i(t) = \sum_j (i/N) N_j(t)$. This isn't strictly true however, because as we observed, Δx_i will be constant during each period, which means we must evaluate N_j at the start of said period. To make this clear, define $\tilde{x}(t) = x(t = T_{n-1})$ for $T_{n-1} < t < T_n$. The tilde notation signifies that throughout the n^{th} period the quantity x is fixed at its value at the start of that period. In terms of this tilde notation, the desired result is $L_i = \sum_j (i/N)\tilde{N}_j(t) = i\tilde{c}$.

To find the firing rate, we follow the procedure used in the main text for the Scramblers: we decompose the firing rate into two parts: $R_i = R_i^0 - R_i^a$. The rate of firing in the absence of absorption will be $R_i^0 = \tilde{c}_i v_i = \tilde{c}_i(1 + v_{\text{pulse}})$. The absorption rate will again be $R_i^a = \sum_j (j/N)\tilde{N}_i R_j = \tilde{c}_i \sum_j j R_j$.

We next determine v_{pulse}. The "pulse velocity" due to a cluster of size j will be (absolute number of pulses per sec) $\times$ (distance per pulse). Since R_j is the rate of firing of c_i, $R_j N$ will give the absolute number of fires. The distance per pulse is j/N. The total pulse velocity is thus $v_{\text{pulse}} = \sum_j (N R_j)(j/N) = \sum_j j R_j$. This gives $R_i^0 = \tilde{c}_i(1 + \sum_j j R_j)$. Putting all this together, we find

$$R_i = \tilde{c}_i(1 + \sum_j j R_j) - \tilde{c}_i \sum_j j R_j \qquad (8)$$

which reduces to $R_i = \tilde{c}_i$.

Substituting our expressions for R_i and L_i into the rate equation for c then yields

$$\dot{c} = -\sum_i L_i R_i = -\sum_i i\tilde{c}\tilde{c}_i = -\tilde{c}\sum_i i\tilde{c}_i = -\tilde{c} \qquad (9)$$

and hence $\dot{c} = -\tilde{c}$. Thus, we see that $\dot{c}$ will be a piecewise linear function.

To solve for this function, we need to determine the periods $\{T_n\}$. The average speed is $v = 1 + v_{\text{pulse}} = 1 + \sum_j j R_j = 1 + \sum_j j c_j = 2$. This gives $\{T_n\} = \{0, 0.5, 1, \dots\}$. For these values of $\{T_n\}$ and the initial condition $c(0) = 1$, the solution of Eq. (9) is the piecewise linear function

$$c(t) = \frac{p + 2 - 2t}{2^{p+1}} \quad \text{for} \quad \frac{p}{2} < t < \frac{p+1}{2}, \qquad (10)$$

where $p = 0, 1, 2, \dots$. Hence, for the Firefly model, the disorder parameter $c(t)$ is a series of line segments of length 0.5, with slopes that are sequentially reduced by a factor of 2.

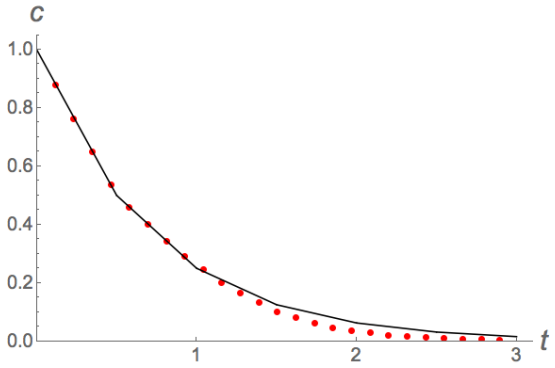

FIG. 5: Piecewise linear behavior of the disorder parameter $c(t)$ for the Firefly model. Black curve, theoretical prediction (10); red dots, simulation results for $N = 5000$ oscillators.

We stress however that our results for $L_i, R_i, v_{\text{pulse}}$, and $c(t)$ are only leading-order approximations, based on mean-field arguments. We expect fluctuations around these values. Our hope is that these will be small.

Figure 5 shows the simulated behavior of $c(t)$ against the mean-field prediction (10). As can be seen, there is reasonable agreement until late times.

Now that we have $c(t)$, the next target is $c_i(t)$. Carrying out the same analysis as for the Scramblers, we find

$$\dot{c}_i = -2\tilde{c}_i + \sum_{k=1}^{i} \tilde{c}_k \, e^{-k\tilde{c}} \sum_{\sum p a_p = i-k} \left(\prod_{p \geq 1} \frac{(k\tilde{c}_p)^{a_p}}{a_p!} \right). \qquad (11)$$

Since the quantities on the right hand side are held fixed over each period, solving for each c_i is straightforward.

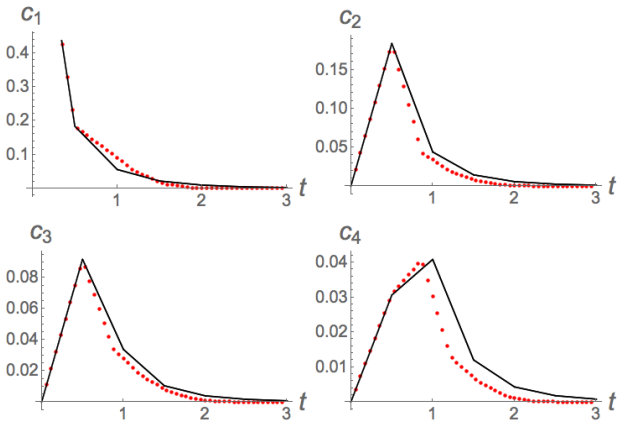

FIG. 6: Time evolution of the cluster densities c_1 through c_4 for the Firefly model with uniformly random initial conditions. Black curves, theoretical predictions derived from Eq. (11); red dots; simulation results for $N = 3 \times 10^4$ oscillators. The discrepancies are due to finite-N effects.

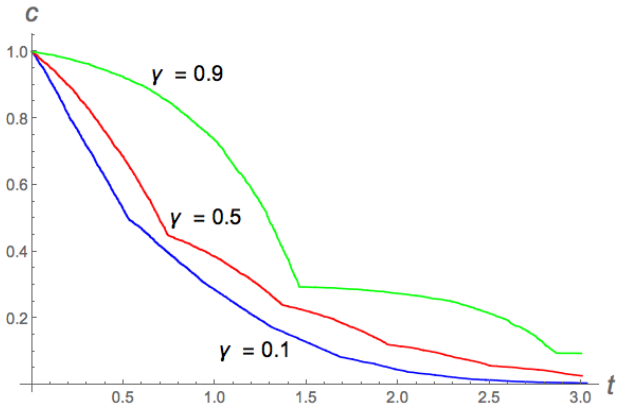

FIG. 7: Time evolution of the disorder parameter c for the Firefly model with uniformly random initial conditions and concave-down charging curve, for three values of the concavity parameter γ. Simulation results are shown for $N = 10^4$ oscillators and pulse size $\epsilon = 1/N$.

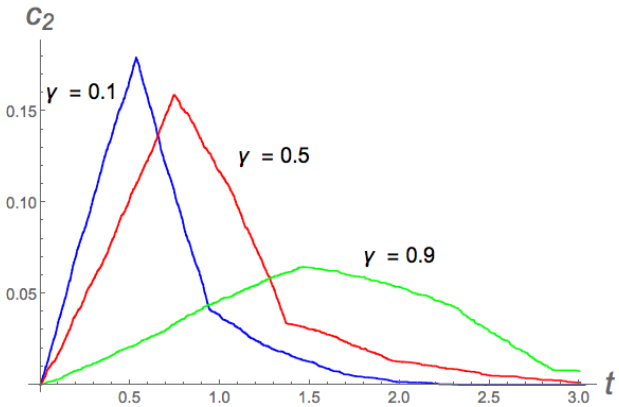

FIG. 8: Time evolution of c_2, the density of 2-clusters, for the Firefly model with uniformly random initial conditions and concave-down charging curve, for three values of the concavity parameter γ. Simulation results are shown for $N = 10^4$ oscillators and pulse size $\epsilon = 1/N$.

Figure 6 shows the resulting solutions along with simulated values. Reasonable agreement is evident. We restate that our results are mean-field equations, so some discrepancy is expected. The moments $M_i(t)$ will also be piecewise linear, and can be obtained in a similarly straightforward manner, following the methods shown in the main text.

Nonlinear charging curve

Until now in our treatment of the deterministic Firefly model, we have assumed that the oscillators rise linearly to threshold. This assumption is valid for the electronic oscillators used in sensor networks, but for neurons and cardiac pacemaker cells, a nonlinear rise to threshold is more appropriate. For this reason, and also on mathematical grounds, it is natural to ask how a nonlinear charging curve would affect the transient dynamics.

To do so, we return to the traditional Peskin model [1, 2]. Its voltage dynamics are governed by $\dot{x}_i = S_0 - \gamma x_i$, where the parameters $S_0 > 0$ and $\gamma \geq 0$. When an oscillator fires, it kicks all other oscillators up by ϵ or up to threshold, whichever is less. For γ strictly greater than zero, the system is guaranteed to end up with all oscillators firing in unison, as proven in Ref. [2], but almost nothing is known about the model's cluster dynamics en route to synchrony. The analysis becomes much more diffult when concavity is included, for reasons discussed in the main text; in short, one can no longer assume that the oscillators are uniformly distributed in voltage at all times. Hence we numerically explore the effect of concavity. We find that for small nonlinearity $\gamma > 0$, the cluster dynamics are similar to what we have already discussed for the Scrambler and linear Firefly models.

Figure 7 shows that for the lowest value of γ, the decay curve for c closely resembles that shown in Figure 5 for the linear Firefly model. The same is true for the individual cluster densities; compare, for example, the curves for c_2 in Figure 8 and Figure 6. For higher values of γ, the cluster dynamics show new effects, not yet understood theoretically. Although these results are preliminary, they suggest that the simplified Scrambler and linear Firefly models are insensitive to small amounts of concavity in the charging curve. In this sense, these models provide reasonable approximations to systems with charging curves that are concave down, the case most often studied in the literature.

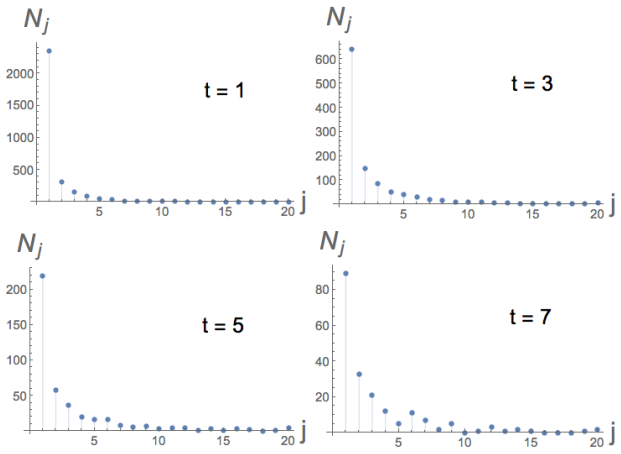

FIG. 9: Simulated N_j versus j for $N = 5000$ oscillators at various times. N_j decreases (roughly) monotonically with increasing cluster size j at every t.

IV. BREAKDOWN OF APPROXIMATIONS

When deriving the rate equations for c_i, we made the assumption that $N_i \gg 1$ for all i (recall that N_i denotes the number of clusters of size i). This assumption let us define c_i in terms of ensemble averages, $c_i := \langle N_i \rangle / N$, which in turn let us use probabilistic arguments in our analysis. This assumption clearly cannot be satisfied for all cluster sizes i. For instance, when $t \to \infty$, we know just one giant cluster of size N remains: $N_N = 1$ and $N_i = 0$ for all other i.

It is not entirely clear how to estimate the time T at which our assumptions break down, but one can obtain an upper bound based on when $c(T) \approx 1/N$, at which time almost the whole system has coalesced into a giant synchronized cluster. Combining this criterion with our chief result that $c(t) = \exp(-t/2)$, we get $T \lesssim 2\ln(N)$. Thus the breakdown time scales at most logarithmically with the size of the system.

What effect does this breakdown of our assumptions have on the predicted cluster kinetics? We examine this issue by simulating the Scrambler model for $N = 5000$ oscillators, and computing N_j for $j \leq 20$ at various times. The results are shown in Figure 9.

The first observation is that N_j decreases (roughly) monotonically with increasing cluster size at every t. Smaller clusters are more abundant than larger ones for the times shown; $N_j \gg N_k$ for $j \ll k$. Our assumption that $N_j \gg 1$ therefore gets worse as the cluster size j increases.

The second observation is that the above is true for much of the synchronization process. For systems of size $N = 5000$, the total time elapsed until complete synchronization occurs is typically $t \approx 10$. But $c(t)$ has typically fallen to ≈ 0.1 by $t = 5$, and to ≈ 0.01 by $t = 7$. So for $t \lesssim 7$, our assumption that $N_j \gg 1$ is good for small clusters, and gets worse for larger clusters. This means our analytical results for c_j should correspondingly worsen for increasing j. As can be seen in Figure 10, this is indeed the case.

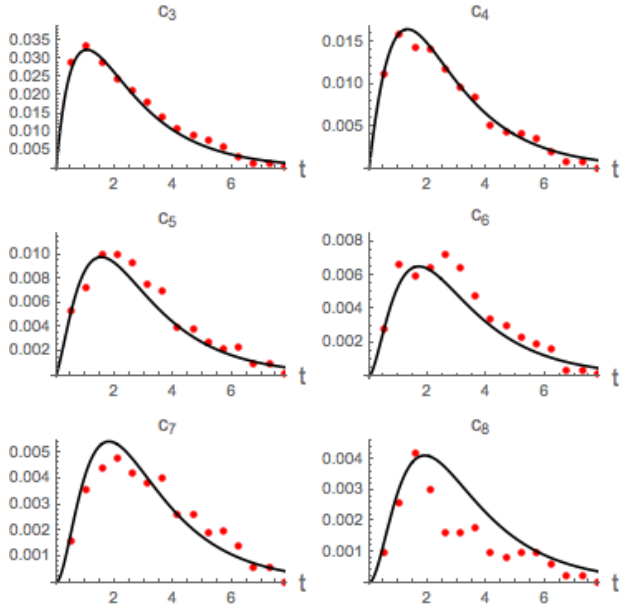

FIG. 10: Theoretical and simulated cluster densities $c_3(t)$ through $c_8(t)$. Solid line show theoretical predictions computed from numerically integration of equation (3). Data points show simulation results for $N = 5000$ oscillators. As expected, the agreement between theory and simulation gets steadily worse with increasing cluster size.

These observations explain why the mean-field approximation for $c(t)$ is good for so long. By definition, $c(t) = \sum_i c_i(t)$. Because $c_j \gg c_k$ for $j \ll k$ (which follows from $N_j \gg N_k$ for $j \ll k$), $\sum_i c_i$ is dominated by small cluster sizes i, for which the theory is most accurate.

Final stages of synchronization

For $t \gtrsim 7$ (again, for a system of size $N = 5000$), only a few large clusters remain. Our assumption $N_j \gg 1$ is then clearly violated, and so the mean-field results no longer apply. In the final stages of the synchronization process, then, we expect large deviations from the mean-field results. To verify this, we calculated the relative errors shown in Figure 11 below.

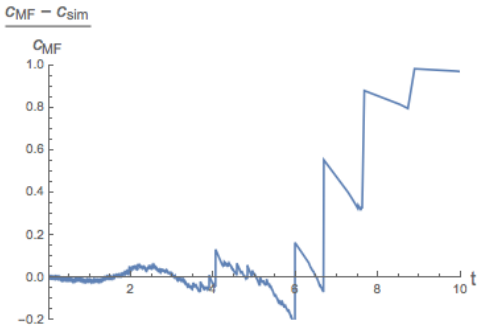

FIG. 11: Relative error between mean-field prediction and simulated $c(t)$, for $N = 5000$ oscillators.

Around $t \approx 7$ the error starts to climb. In particular, we see $c_{MF} > c_{sim}$, meaning the synchronization of the simulated system is faster than the mean-field prediction. As $t \to \infty$ the process becomes more and more stochastic, whereby c_{sim} oscillates noisily around c, until it drops stochastically to zero, as we show in Figure 12.

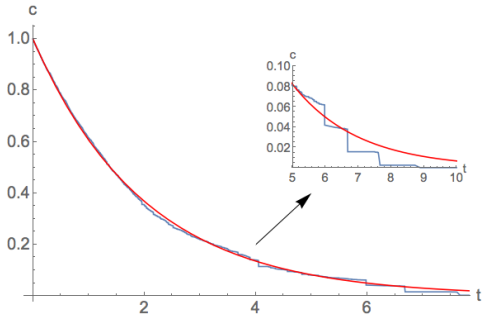

FIG. 12: Mean-field prediction (red curve) and simulated $c(t)$ (blue curve) for $N = 5000$ oscillators. As can be seen in the inset, the simulation results get noisy for large t.

[1] C. S. Peskin, *Mathematical Aspects of Heart Physiology* (Courant Institute of Mathematical Sciences, New York, 1975), pp. 268–278.

[2] R. E. Mirollo and S. H. Strogatz, SIAM Journal on Applied Mathematics **50**, 1645 (1990).

[3] S. Bottani, Phys. Rev. Lett. **74**, 4189 (1995).