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008

JUMP BIFURCATION AND HYSTERESIS IN AN INFINITE-DIMENSIONAL DYNAMICAL SYSTEM OF COUPLED SPINS*

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Abstract. This paper studies an infinite-dimensional dynamical system that models a collection of coupled spins in a random magnetic field. A state of the system is given by a self-map of the unit circle. Critical states for the dynamical system correspond to equilibrium configurations of the spins. Exact solutions are obtained for all the critical states and their stability is characterized by analyzing the second variation of the system's potential function at those states. It is proven rigorously that the system exhibits a jump bifurcation and hysteresis as the coupling between spins is varied.

Key words. bifurcation, dynamical system, phase transition, spin system

AMS(MOS) subject classifications. 58F14, 82A57, 34F05

1. Introduction. There is currently a great deal of interest in the collective behavior of dynamical systems with many interacting subunits [2]. Such systems are very difficult to analyze rigorously unless some simplifying assumptions are made. "Phase models," in which each subunit is characterized by a single phase angle θ_i, provide an analytically tractable class of many-body systems. Phase models have been used in a wide variety of scientific contexts, including charge-density wave transport [8], [9], [23], [24], magnetic spin systems [3], [26], oscillating chemical reactions [15], [18], [28], networks of biological oscillators [4], [7], [14], [27], [28], and arrays of Josephson junctions [11], [25].

One of the key themes in these studies is the competition between order and disorder. Typically the system behaves incoherently if the coupling between subunits is too weak. The incoherence may be due to thermal noise, random fields, inpurities, or other inhomogeneities, depending on the problem. As the coupling between the subunits is increased, the system may undergo a phase transition from a disordered to an ordered state. For example, there has been much recent work on the sudden onset of self-synchronization in populations of coupled oscillators with randomly distributed [5], [6], [16], [19], [21], [22], [27], [28] or fluctuating [15], [20] intrinsic frequencies.

In this paper we assume that the disorder arises from a different mechanism, known as a random pinning field. This random field tries to pin each θ_i at a random angle α_i. It is opposed by an attractive interaction between the phases, which favors a uniform alignment with $\theta_i = \theta_j$ for all i, j. More specifically, consider the nonlinear dynamical system

$$\dot{\theta}_i = \sin(\alpha_i - \theta_i) + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \cdots, N. \tag{1.1}$$

Here the α_i are independent random variables uniformly distributed on $[0, 2\pi]$ and $K \geqq 0$ is a parameter. We wish to understand the stable critical points of this system, and how they change qualitatively as K ranges from 0 to ∞.

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The system (1.1) arises in condensed matter physics as a simple model of a random magnet at zero temperature [3], [13], [26], [30]. The magnet is imagined to be composed of a large number of spins $s_j = e^{i\theta_j}$, modeled as unit vectors in the plane. Because the system is assumed to be at zero temperature, it obeys gradient dynamics:

$$\dot{\theta}_i = -\partial H/\partial \theta_i \tag{1.2}$$

where the potential function H satisfies

$$H = -\sum_{i=1}^{N} \cos(\alpha_i - \theta_i) - \frac{K}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \cos(\theta_i - \theta_j). \tag{1.3}$$

In other words, the system lowers its energy by flowing down the energy gradient in configuration space. As $t \to \infty$, the system approaches some critical point of (1.3). At a *stable* critical point, the spins adopt a configuration that locally minimizes H.

Thus to characterize the long-time behavior of the system, it suffices to study the critical points of (1.3). In equilibrium the system balances two competing terms in (1.3). The $\cos(\alpha_i - \theta_i)$ term favors an incoherent arrangement of spins with $\theta_i = \alpha_i$ for all i, where the α_i are fixed random angles. Physically this term models a local magnetic field of unit strength and random direction α_i. This random field is opposed by the terms $\cos(\theta_i - \theta_j)$ in (1.3), which favor a coherent configuration with all the spins aligned ($\theta_i = \theta_j$ for all i, j). This tendency to align is due to the ferromagnetic coupling among the spins. In the approximation studied here, the coupling is infinite-range; all spins interact equally with a strength K/N. The parameter K measures the coupling strength (relative to the random field strength, here scaled to unity), and the factor $1/N$ keeps the total field on any spin bounded as $N \to \infty$.

When $K = 0$, the stable configuration is clearly the incoherent state $\theta_i = \alpha_i$ for all i, whereas for $K \to \infty$, the stable configuration is the coherent state $\theta_i = \theta_j$ for all i, j. To quantify the transition from incoherence to coherence as K runs from 0 to ∞, we introduce the complex order parameter

$$r e^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \tag{1.4}$$

which represents the average spin or magnetization of the system (Fig. 1). In (1.4), $r = 0$ corresponds to incoherence and $r = 1$ corresponds to perfect alignment. The angle $\psi \in [0, 2\pi]$ is well defined for $r > 0$ and measures the average phase of the spin configuration.

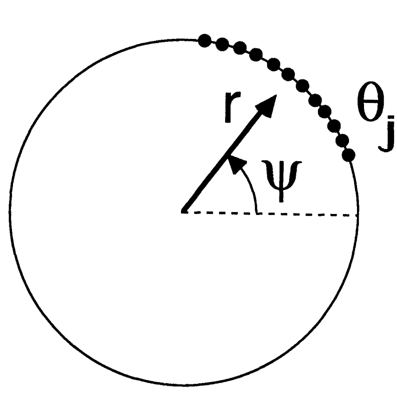

FIG. 1. *The order parameter* $r e^{i\psi} = N^{-1} \sum_j e^{i\theta_j}$. *Its modulus r measures the coherence of the spin configuration and the angle* ψ *measures the average phase.*

We are interested in the behavior of (1.1) as $N \to \infty$, so we make the following infinite-dimensional dynamical system our primary object of study:

$$\dot{\theta}_\alpha = \sin(\alpha - \theta_\alpha) + \frac{K}{2\pi} \int_0^{2\pi} \sin(\theta_\beta - \theta_\alpha)\, d\beta. \tag{1.5}$$

Now the states of the system are *continuous*, 2π-periodic real functions $\theta : \alpha \mapsto \theta_\alpha$, which should be thought of as self-maps of the unit circle. (There is in fact little to be gained by including discontinuous functions in the space of admissible states, so we discuss these in an appendix.) The system (1.5) is also a gradient flow with potential given by the continuous version of (1.3):

$$H = -\frac{1}{2\pi} \int_0^{2\pi} \cos(\alpha - \theta_\alpha)\, d\alpha - \frac{K}{2} \cdot \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \cos(\theta_\alpha - \theta_\beta)\, d\alpha\, d\beta. \tag{1.6}$$

(Note that $\dot{\theta}_\alpha = -2\pi(\partial H / \partial \theta_\alpha)$; the extra factor of 2π in this definition of H proves convenient later.) As $N \to \infty$, the order parameter (1.4) becomes

$$r\, e^{i\psi} = \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta_\alpha}\, d\alpha. \tag{1.7}$$

This paper concerns the following question: For the critical states of (1.5), how does the coherence r depend on the coupling strength K? To answer this question, we first exhibit all states satisfying the critical condition $\nabla H = 0$. Then we calculate the index of the second variation of H at these states. In particular, we will calculate precisely all stable critical states. We will find that there are two families of critical states, one with $r > 0$ and one with $r = 0$. The graph of r versus K is shown in Fig. 2. Here $K_c \approx 1.489$, $r_c \approx 0.739$. Thus for a given value of K there are at most three values of r ($r = 0$ is always a solution) for which a critical state with that value of r exists. The stable critical states (corresponding to heavy lines in Fig. 2) are those with $r = 0$ for $0 < K < 2$, and $r > r_c$ for $K > K_c$. The system undergoes jump bifurcations at $K = 2$ and $K = K_c$.

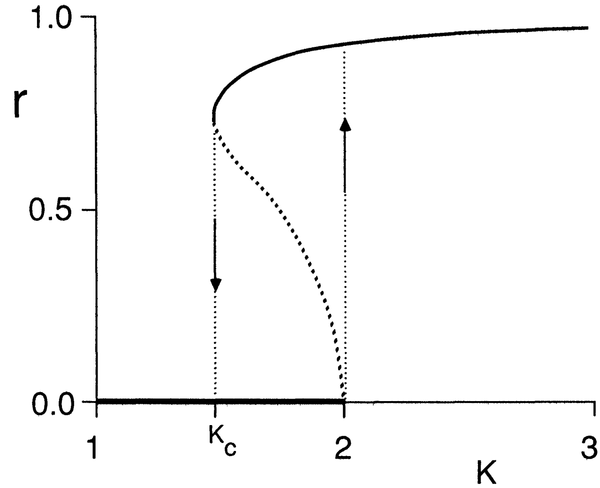

FIG. 2. *Coherence r plotted against the coupling strength K, for critical states of the dynamical system* (1.5). *Heavy lines, locally stable states; thin or broken lines, unstable states. The system exhibits jump bifurcations and hysteresis at $K = K_c$ and $K = 2$.*

The system (1.5) therefore exhibits hysteresis in the range $K_c \leqq K \leqq 2$: if K is small, the system will evolve from any initial state to a unique stable critical state with $r=0$. If K is increased, this state remains critical and stable until $K>2$, when the system will jump up to a new stable state with $r>0$ as shown in Fig. 2. If K is then decreased below two, this state will change but r will remain $>r_c$ until $K=K_c$, then drop to the $r=0$ stable state when $K<K_c$.

This paper is organized as follows. We discuss the critical states in § 2, then analyze them for stability in § 3. Section 4 contains a discussion of our results in relation to physics. Appendix A is devoted to discontinuous states, and Appendix B concerns an elliptic integral that arises in the course of our calculations.

2. Critical states.

2.1. Governing equation for critical states. The critical states of (1.5) are functions $\alpha \mapsto \theta_\alpha$ whch satisfy

$$0=\sin(\alpha-\theta_\alpha)+\frac{K}{2\pi}\int_0^{2\pi}\sin(\theta_\beta-\theta_\alpha)\,d\beta. \tag{2.1}$$

Using the definition (1.7) of the order parameter $r\,e^{i\psi}$, we can more conveniently express the integral in (2.1):

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(\theta_\beta-\theta_\alpha)\,d\beta=r\sin(\psi-\theta_\alpha). \tag{2.2}$$

Hence the condition for a critical state becomes

$$0=\sin(\alpha-\theta_\alpha)+rK\sin(\psi-\theta_\alpha). \tag{2.3}$$

The system (1.5) is invariant under the following one-parameter group of transformations:

$$\theta_\alpha[\gamma]=\theta_{\alpha+\gamma}-\gamma. \tag{2.4}$$

In fact, the potential (1.6) can be rewritten

$$H=-\frac{1}{2\pi}\int_0^{2\pi}\cos(\alpha-\theta_\alpha)\,d\alpha-\frac{1}{2}r^2K, \tag{2.5}$$

because

$$\begin{aligned}&\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\cos(\theta_\beta-\theta_\alpha)\,d\alpha\,d\beta\\&=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}(\cos\theta_\alpha\cos\theta_\beta+\sin\theta_\alpha\sin\theta_\beta)\,d\alpha\,d\beta\\&=\frac{1}{(2\pi)^2}\left[\left(\int_0^{2\pi}\cos\theta_\alpha\,d\alpha\right)^2+\left(\int_0^{2\pi}\sin\theta_\alpha\,d\alpha\right)^2\right]=r^2.\end{aligned}$$

Clearly, the potential H is invariant under the transformation $\theta_\alpha \to \theta_\alpha[\gamma]$. Physically, this invariance expresses the fact that the energy of a configuration is unchanged if all the angles α_i and θ_i are rotated by the same amount. One nice consequence of this is that in our search for critical states, we may assume without loss of generality that $\psi=0$. We will discuss another consequence of this invariance when we study the stability of critical states.

We have now arrived at the following condition for critical states:

$$0=\sin(\alpha-\theta_\alpha)-rK\sin\theta_\alpha \tag{2.6a}$$

where

$$r = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta_\alpha}\, d\alpha \geqq 0. \tag{2.6b}$$

Note that (2.6a, b) are *necessary and sufficient* conditions for a state θ_α to be critical (assuming $\psi = 0$ when $r > 0$).

2.2. Parametrization of critical states. From now on, let

$$u = rK. \tag{2.7}$$

It turns out that the critical states are most easily described in terms of the variable u.

When $u = 0$ ($\Leftrightarrow r = 0$) the only continuous solutions to (2.6a) are

$$\theta_\alpha = \alpha \quad \text{or} \quad \theta_\alpha = \alpha + \pi. \tag{2.8}$$

Hence these are the only critical states with $r = 0$. Note that these critical states exist for every K.

When $u > 0$, we must solve

$$\sin(\alpha - \theta_\alpha) = u \sin\theta_\alpha. \tag{2.9}$$

Write

$$\sin(\alpha - \theta_\alpha) = \frac{e^{i(\alpha-\theta_\alpha)} - e^{-i(\alpha-\theta_\alpha)}}{2i}, \qquad \sin\theta_\alpha = \frac{e^{i\theta_\alpha} - e^{-i\theta_\alpha}}{2i}.$$

Then (2.9) becomes

$$e^{i(\alpha-\theta_\alpha)} - e^{-i(\alpha-\theta_\alpha)} = u(e^{i\theta_\alpha} - e^{-i\theta_\alpha}). \tag{2.10}$$

Solving for $e^{i\theta_\alpha}$ gives

$$e^{2i\theta_\alpha} = \frac{u + e^{i\alpha}}{u + e^{-i\alpha}} = \frac{(u + e^{i\alpha})^2}{|u + e^{i\alpha}|^2}. \tag{2.11}$$

If $u \neq 1$, the right-hand side has exactly two continuous square roots, given by

$$e^{i\theta_\alpha} = \pm\frac{u + e^{i\alpha}}{|u + e^{i\alpha}|}. \tag{2.12}$$

Since we require $r > 0$, we must in fact have

$$e^{i\theta_\alpha} = \frac{u + e^{i\alpha}}{|u + e^{i\alpha}|}. \tag{2.13}$$

This relation between α, u, and θ_α is illustrated by the geometric construction shown in Fig. 3.

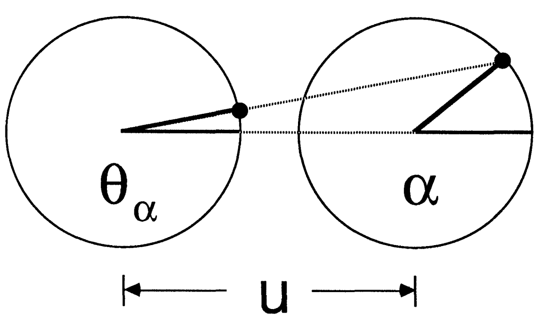

FIG. 3. *The position of each spin $e^{i\theta_\alpha}$ in the critical state* (2.13) *is determined by $e^{i\alpha}$ and by u $(=Kr)$. In the geometric construction shown here, $\psi = 0$ and the circles have unit radius.*

When $u=1$, $(1+e^{i\alpha})/(1+e^{-i\alpha})=e^{i\alpha}$, so $\theta_\alpha=\alpha/2$ or $\alpha/2+\pi$; hence there are *no* continuous solutions. (However, see Appendix A.)

Now, fix some $K>0$. In order for the state (2.13) to be critical with this value of K, we need

$$r=\frac{1}{2\pi}\int_0^{2\pi} e^{i\theta_\alpha}\,d\alpha,$$

which implies

$$\frac{u}{K}=\frac{1}{2\pi}\int_0^{2\pi}\frac{u+e^{i\alpha}}{|u+e^{i\alpha}|}\,d\alpha. \tag{2.14}$$

Let $f(u)$ denote the integral in (2.14):

$$f(u)=\frac{1}{2\pi}\int_0^{2\pi}\frac{u+e^{i\alpha}}{|u+e^{i\alpha}|}\,d\alpha. \tag{2.15}$$

Then all the critical states with $r>0$, $\psi=0$ are parametrized as follows: For any $u>0$, $u\neq 1$, let

$$e^{i\theta_\alpha}=\frac{u+e^{i\alpha}}{|u+e^{i\alpha}|};\qquad K=u/f(u);\qquad r=f(u). \tag{2.16}$$

To graph r as a function of K, we must sketch the parametric curve $K=u/f(u)$, $r=f(u)$. First we must study the function $f(u)$. (In Appendix B we show that $f(u)$ is an elliptic integral; here we prefer an elementary approach.)

2.3. Properties of $f(u)$. For $0\leqq u<+\infty$

$$f(u)=\frac{1}{2\pi}\int_0^{2\pi}(u+\cos\alpha)(1+u^2+2u\cos\alpha)^{-1/2}\,d\alpha. \tag{2.17}$$

$$f(0)=0,\qquad f(+\infty)=1, \tag{2.18}$$

$$f(1)=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1+\cos\alpha}{2}\right)^{1/2}d\alpha=\frac{2}{\pi}. \tag{2.19}$$

The function $f(u)$ is continuous, and for $u\neq 1$

$$f'(u)=\frac{1}{2\pi}\int_0^{2\pi}\sin^2\alpha(1+u^2+2u\cos\alpha)^{-3/2}\,d\alpha. \tag{2.20}$$

Hence $f(u)$ is *increasing*. Moreover $\lim_{u\to 1}f'(u)=+\infty$ so the graph of $f(u)$ has a vertical tangent at $u=1$. Note also that $f'(0)=\frac{1}{2}$.

We also need to understand the concavity of the graph of $f(u)$. To do this, we expand $f(u)$ in a power series. Assume first that $0\leqq u<1$. Then

$$\left(\frac{u+e^{i\alpha}}{u+e^{-i\alpha}}\right)^{1/2}=e^{i\alpha}\left(\frac{1+u\,e^{-i\alpha}}{1+u\,e^{i\alpha}}\right)^{1/2}. \tag{2.21}$$

We expand the integrand of (2.15) using the binomial series

$$(1+x)^r=\sum_{k=0}^{\infty}\binom{r}{k}x^r,$$

getting

$$\left(\frac{u+e^{i\alpha}}{u+e^{-i\alpha}}\right)^{1/2}=e^{i\alpha}\sum_{k=0}^{\infty}\binom{1/2}{k}u^k e^{-ik\alpha}\sum_{j=0}^{\infty}\binom{-1/2}{j}u^j e^{-ij\alpha}. \tag{2.22}$$

Since $1/2\pi \int_0^{2\pi} e^{ik\alpha}\, d\alpha = 0$ unless $k=0$, we get from (2.15) and (2.22)

$$f(u) = \sum_{k=0}^{\infty} \binom{-1/2}{k}\binom{1/2}{k+1} u^{2k+1} \quad \text{for } 0 \leqq u < 1. \tag{2.23}$$

Let

$$a_k = \binom{-1/2}{k}\binom{1/2}{k+1}. \tag{2.24}$$

Then

$$a_0 = \frac{1}{2} \quad \text{and} \quad a_{k+1} = \frac{(k+1/2)^2}{(k+1)(k+2)} a_k \quad \text{for } k \geqq 0.$$

Hence $a_k > 0$ for all k. In particular, $f''(u) > 0$ and the graph is concave up for $0 \leqq u < 1$.

For $u > 1$ we proceed similarly. Write

$$\left(\frac{u+e^{i\alpha}}{u+e^{-i\alpha}}\right)^{1/2} = (1+u^{-1}e^{i\alpha})^{1/2}(1+u^{-1}e^{-i\alpha})^{-1/2}. \tag{2.25}$$

Then as before, we expand the right-hand side, integrate, and get

$$f(u) = \sum_{k=0}^{\infty} \binom{1/2}{k}\binom{-1/2}{k} u^{-2k} \quad \text{for } u > 1. \tag{2.26}$$

Let

$$b_k = \binom{1/2}{k}\binom{-1/2}{k}. \tag{2.27}$$

Then $b_0 = 1$, $b_1 = -1/4$, and $b_{k+1} = (k-1/2)(k+1/2)/(k+1)^2$ for $k \geqq 0$. Hence $b_k < 0$ for $k \geqq 1$. In particular, $f''(u) < 0$ and the graph of f is concave down for $u > 1$.

2.4. Graphs. We illustrate the graph of $f(u)$ in Fig. 4.

We let (u_c, r_c) be the *unique* point on the curve satisfying the relation

$$f'(u_c) = \frac{f(u_c)}{u_c} \tag{2.28}$$

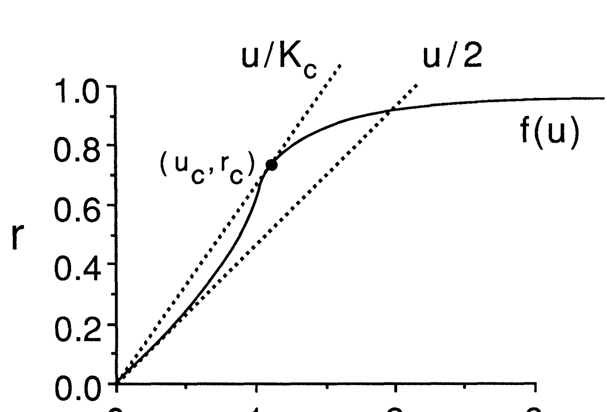

FIG. 4. *The graph of the function $f(u)$ defined by* (2.17). *Intersections of $f(u)$ with the lines u/K give the points (K, r) on the curve* (2.31). *Dashed lines correspond to jump bifurcations at $K = K_c$ and $K = 2$.*

(i.e., the tangent line passes through the origin). Let

$$K_c = u_c/f(u_c). \tag{2.29}$$

Then

$$K_c \approx 1.489, \quad r_c \approx 0.739, \quad u_c \approx 1.100, \tag{2.30}$$

as determined by numerical integration of $f(u)$. More specifically, the critical value K_c in (2.30) was estimated by minimizing $u/f(u)$ numerically. The minimum occurs when $u = u_c$, and then $f(u_c)$ gives r_c.

Now it is easy to graph the curve

$$K = u/f(u), \qquad r = f(u). \tag{2.31}$$

The resulting curve was shown earlier in Fig. 2.

As $u \to 0^+$, $K \to 1/f'(0) = 2$. Hence the curve branches from $r = 0$ at the value $K = 2$. Furthermore

$$f'(u) > 0 \quad \text{and} \quad (u/f(u))' = \frac{uf'(u) - f(u)}{f(u)^2}$$

so

$$(u/f(u))' < 0 \quad \text{for } 0 < u < u_c,$$
$$(u/f(u))' > 0 \quad \text{for } u > u_c.$$

The graph of r against K therefore looks like that shown earlier in Fig. 2.

2.5. Summary. In summary, for a given value of K there are at most three families of critical states for the system (1.5), as shown in Fig. 2. The critical states fall into two categories, depending on the value of r:

$$\text{(I)} \qquad \theta_\alpha = \alpha \quad \text{or} \quad \theta_\alpha = \alpha + \pi. \quad \text{Here } r = 0. \tag{2.32}$$

$$\text{(II)} \qquad e^{i\theta_\alpha} = \frac{rK + e^{i\alpha}}{|rK + e^{-i\alpha}|}, \qquad r > 0, \tag{2.33}$$

where (r, K) is a point on the graph shown in Fig. 2, or transforms of this solution (II) by the group action (2.4). There may be 0, 1, or 2 families of this form. They are topologically nonintersecting circles in the space of all states.

3. Stability.

3.1. Computing the second variation. Our next task is to analyze the critical states found in § 2 for stability. Recall that our system has a potential (1.6) given by

$$H(\theta) = -\left[\frac{1}{2\pi}\int_0^{2\pi} \cos(\alpha - \theta_\alpha)\, d\alpha + \frac{K}{2}\cdot\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \cos(\theta_\beta - \theta_\alpha)\, d\alpha\, d\beta\right]$$

where $\theta : \alpha \mapsto \theta_\alpha$ is a continuous, 2π-periodic function. Let ϕ be any other continuous, 2π-periodic function, and let

$$\theta_\alpha(\varepsilon) = \theta_\alpha + \varepsilon\phi_\alpha. \tag{3.1}$$

We want to examine the quadratic form

$$\Gamma_\theta(\phi) =_{\text{def}} \frac{d^2}{d\varepsilon^2} H(\theta(\varepsilon))\bigg|_{\varepsilon=0}. \tag{3.2}$$

Differentiating gives

$$\Gamma_\theta(\phi)=\frac{1}{2\pi}\int_0^{2\pi}\cos(\alpha-\theta_\alpha)\phi_\alpha^2\,d\alpha + \frac{K}{2}\cdot\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\cos(\theta_\beta-\theta_\alpha)(\phi_\beta-\phi_\alpha)^2\,d\alpha\,d\beta. \tag{3.3}$$

To simplify (3.3) we expand the second integral:

$$\begin{aligned}\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}&\cos(\theta_\beta-\theta_\alpha)(\phi_\beta-\phi_\alpha)^2\,d\alpha\,d\beta\\ &=2\cdot\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\cos\theta_\beta\cos\theta_\alpha\,\phi_\alpha^2\,d\alpha\,d\beta\\ &\quad+2\cdot\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\sin\theta_\beta\sin\theta_\alpha\,\phi_\alpha^2\,d\alpha\,d\beta\\ &\quad-2\cdot\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(\cos\theta_\beta\cos\theta_\alpha+\sin\theta_\beta\sin\theta_\alpha)\phi_\alpha\phi_\beta\,d\alpha\,d\beta.\end{aligned} \tag{3.4}$$

We assume without loss of generality that $1/2\pi\int_0^{2\pi}e^{i\theta_\alpha}\,d\alpha=r\geqq 0$. Then (3.4) reduces to

$$\begin{aligned}\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}&\cos(\theta_\beta-\theta_\alpha)(\phi_\beta-\phi_\alpha)^2\,d\alpha\,d\beta\\ &=2r\cdot\frac{1}{2\pi}\int_0^{2\pi}(\cos\theta_\alpha)\phi_\alpha^2\,d\alpha-2\left|\frac{1}{2\pi}\int_0^{2\pi}e^{i\theta_\alpha}\phi_\alpha\,d\alpha\right|^2.\end{aligned} \tag{3.5}$$

Hence

$$\begin{aligned}\Gamma_\theta(\phi)=&\frac{1}{2\pi}\int_0^{2\pi}[\cos(\theta_\alpha-\alpha)+rK\cos\theta_\alpha]\phi_\alpha^2\,d\alpha\\ &-K\left|\frac{1}{2\pi}\int_0^{2\pi}e^{i\theta_\alpha}\phi_\alpha\,d\alpha\right|^2.\end{aligned} \tag{3.6}$$

As in § 2, let

$$u=rK,\qquad u\geqq 0. \tag{3.7}$$

We want to understand as much as possible about the quadratic form $\Gamma_\theta(\phi)$ for the different critical states θ. In particular, we want to know when $\Gamma_\theta(\phi)\geqq 0$, i.e., when the state θ is locally stable. Fortunately, we shall be able to completely diagonalize Γ_θ in a certain Hilbert space, and calculate precisely all the eigenvalues associated to Γ_θ with respect to that inner product.

3.2. Incoherent states ($u=0$). In the case $u=0$, the critical states are given by (2.8): $\theta_\alpha=\alpha$ or $\theta_\alpha=\alpha+\pi$. If $\theta_\alpha=\alpha+\pi$, then (3.6) implies

$$\Gamma_\theta(\phi)=-\left[\frac{1}{2\pi}\int_0^{2\pi}\phi_\alpha^2\,d\alpha+K\left|\int_0^{2\pi}e^{i\alpha}\phi_\alpha\,d\alpha\right|^2\right]<0 \tag{3.8}$$

unless $\phi_\alpha = 0$ so Γ_θ is *negative definite.* Hence $\theta_\alpha = \alpha + \pi$ is a local *maximum* for the potential H, so θ is unstable (in fact all eigenvalues are negative). Physically, this case corresponds to all spins pointing directly *opposite* to the direction of the local magnetic field.

If $\theta_\alpha = \alpha$,

$$\Gamma_\theta(\phi) = \frac{1}{2\pi}\int_0^{2\pi} \phi_\alpha^2 \, d\alpha - K\left|\frac{1}{2\pi}\int_0^{2\pi} e^{i\alpha}\phi_\alpha \, d\alpha\right|^2. \tag{3.9}$$

We work in the Hilbert space $L^2(S^1)$ where the unit circle has measure $d\alpha/2\pi$. Let $\mu_\alpha = \cos\alpha$, $\nu_\alpha = \sin\alpha$. Then $\|\mu\|^2 = \|\nu\|^2 = 1/2$, and $\mu \cdot \nu = 0$. Write ϕ in an orthogonal decomposition using μ, ν:

$$\phi_\alpha = a\frac{\mu_\alpha}{\|\mu\|} + b\frac{\nu_\alpha}{\|\nu\|} + \phi_\alpha^\perp, \tag{3.10}$$

where $\phi^\perp \cdot \mu = \phi^\perp \cdot \nu = 0$. Then

$$\|\phi\|^2 = a^2 + b^2 + \|\phi^\perp\|^2, \tag{3.11}$$

and

$$\Gamma_\theta(\phi) = \|\phi^\perp\|^2 + \left(1 - \frac{K}{2}\right)(a^2 + b^2). \tag{3.12}$$

This equation (3.12) gives an orthogonal decomposition of Γ_θ into two forms. $\|\phi^\perp\|^2$ is positive definite, and depending on K, $(1-(K/2))(a^2+b^2)$ is positive, negative, or 0. Hence

(3.13a) for $K < 2$, Γ_θ is positive definite;

(3.13b) for $K = 2$, Γ_θ is positive definite on a space of codimension 2, and zero on its orthogonal complement;

(3.13c) for $K > 2$, Γ_θ is positive definite on a space of codimension 2, and negative definite on its orthogonal complement.

In particular, $\theta_\alpha = \alpha$ is stable for $K < 2$, unstable for $K > 2$.

3.3. Coherent states ($u > 0$). We can use similar methods to study Γ_θ when $u > 0$. Remember, we do not allow $u = 1$ because it gives rise to a discontinuous critical state (see Appendix A).

From (2.13), the critical states for $u > 0$ are

$$e^{i\theta_\alpha} = \frac{u + e^{i\alpha}}{|u + e^{i\alpha}|}.$$

Then

$$\cos(\theta_\alpha - \alpha) + u\cos\theta_\alpha = |u + e^{i\alpha}|. \tag{3.14}$$

Substituting (3.14) and (2.13) into (3.6) yields

$$\Gamma_\theta(\phi) = \frac{1}{2\pi}\int_0^{2\pi} \phi_\alpha^2 |u + e^{i\alpha}| \, d\alpha - K\left|\frac{1}{2\pi}\int_0^{2\pi} \phi_\alpha \frac{(u + e^{i\alpha})}{|u + e^{i\alpha}|} \, d\alpha\right|^2. \tag{3.15}$$

We now regard $(1/2\pi)|u + e^{i\alpha}| \, d\alpha$ as a measure on S^1. All the following integrals in this section are computed with respect to this measure.

Let μ, $\nu \in L^2(S^1)$ be defined by

$$\mu_\alpha + i\nu_\alpha = \frac{u+e^{i\alpha}}{|u+e^{i\alpha}|^2}. \tag{3.16}$$

Then in $L^2(S^1)$ with this new measure, note that

$$\mu \cdot \nu = 0 \tag{3.17}$$

since μ_α is even in α, ν_α is odd, and $|u+e^{i\alpha}|$ is even. Write

$$\phi_\alpha = a\frac{\mu_\alpha}{\|\mu\|} + b\frac{\nu_\alpha}{\|\nu\|} + \phi_\alpha^\perp \tag{3.18}$$

as before. Then substituting (3.18) into (3.15) gives the result

$$\Gamma_\theta(\phi) = \|\phi^\perp\|^2 + (1-K\|\mu\|^2)a^2 + (1-K\|\nu\|^2)b^2. \tag{3.19}$$

Hence again we have completely decomposed the form Γ_θ into a positive definite form and a form on its orthogonal complement whose eigenvalues can be studied. The problem reduces, therefore, to analyzing $1-K\|\mu\|^2$, $1-K\|\nu\|^2$. We shall show that the eigenvalue $1-K\|\mu\|^2$ vanishes identically and hence the eigenvalue $1-K\|\nu\|^2$ alone determines the stability of the critical states.

First we compute $\|\nu\|^2$:

$$\begin{aligned}\|\nu\|^2 &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin^2\alpha}{|u+e^{i\alpha}|^4}|u+e^{i\alpha}|\,d\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin^2\alpha}{|u+e^{i\alpha}|^3}\,d\alpha.\end{aligned} \tag{3.20}$$

Comparing (3.20) and (2.20) we obtain

$$\|\nu\|^2 = f'(u). \tag{3.21}$$

Therefore (recalling that $K = u/f(u)$)

$$1-K\|\nu\|^2 = 1-\frac{u}{f(u)}f'(u) \tag{3.22}$$

and

$$1-K\|\nu\|^2 > 0 \Leftrightarrow uf'(u) < f(u). \tag{3.23}$$

The stability condition $uf'(u)<f(u)$ means that the tangent line to the curve $y=f(u)$ at the point $(u, f(u))$ meets the y axis above 0 (Fig. 4). From our study of the graph of $f(u)$, we see that this condition is equivalent to $u>u_c$.

Now we turn to the computation of $\|\mu\|^2$:

$$\begin{aligned}\|\mu\|^2 &= \frac{1}{2\pi}\int_0^{2\pi}\frac{(u+\cos\alpha)^2}{|u+e^{i\alpha}|^3}\,d\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{u^2+2u\cos\alpha+\cos^2\alpha}{|u+e^{i\alpha}|^3}.\end{aligned} \tag{3.24}$$

Because $|u+e^{i\alpha}|^2 = u^2+2u\cos\alpha+1$, (3.24) becomes

$$\begin{aligned}\|\mu\|^2 &= \frac{1}{2\pi}\int_0^{2\pi}\frac{|u+e^{i\alpha}|^2-\sin^2\alpha}{|u+e^{i\alpha}|^3}\,d\alpha \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{d\alpha}{|u+e^{i\alpha}|} - \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin^2\alpha}{|u+e^{i\alpha}|^3}\,d\alpha.\end{aligned} \tag{3.25}$$

If we integrate by parts in the second integral, we get

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\sin^2\alpha}{|u+e^{i\alpha}|^3}\,d\alpha=-\frac{1}{u}\,\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos\alpha}{|u+e^{i\alpha}|}\,d\alpha. \tag{3.26}$$

Hence

$$\|\mu\|^2=\frac{1}{u}\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{u+\cos\alpha}{|u+e^{i\alpha}|}\,d\alpha. \tag{3.27}$$

From the definition (2.15) of $f(u)$, (3.27) becomes

$$\|\mu\|^2=\frac{f(u)}{u}. \tag{3.28}$$

Therefore $K\|\mu\|^2=u/f(u)\cdot f(u)/u=1$ for all values of K, so the eigenvalue for the μ direction vanishes identically! Thus

$$1-K\|\mu\|^2\equiv 0. \tag{3.29}$$

The vanishing of the eigenvalue (3.29) may be understood geometrically. It reflects a symmetry in the potential function (2.5). Since the potential for our system is invariant under the one-parameter group action (2.4), the second variation Γ_θ must have at least one null vector, which we can calculate by finding the derivative of the group action at θ. If $e^{i\theta_\alpha}=(u+e^{i\alpha})/(|u+e^{i\alpha}|)$ then

$$\begin{aligned} e^{i\theta_\alpha[\gamma]}&=e^{i\theta_{\alpha+\gamma}}\,e^{-i\gamma}\\ &=e^{-i\gamma}\left(\frac{u+e^{i(\alpha+\gamma)}}{|u+e^{i(\alpha+\gamma)}|}\right). \end{aligned} \tag{3.30}$$

We can calculate the derivative of the group action by differentiating (3.30). This yields

$$\left.\frac{d}{d\gamma}(\theta_\alpha[\gamma])\right|_{\gamma=0}=-u\left(\frac{u+\cos\alpha}{|u+e^{i\alpha}|^2}\right). \tag{3.31}$$

Note that, up to a multiplicative constant, this is the function μ we introduced earlier in (3.16). Therefore the null direction is parallel to μ, and we have

$$1-K\|\mu\|^2=\Gamma_\theta\left(\frac{\mu}{\|\mu\|}\right)=0, \tag{3.32}$$

which confirms (3.29).

3.4. Summary. We now summarize the stability analysis of the critical states. Figure 5 classifies the critical states according to the number of negative eigenvalues (corresponding to unstable directions) and the number of zero eigenvalues (corresponding to neutral directions) for the second variation Γ_θ.

Case 1. $r=0$ (incoherent states). The critical states are $\theta_\alpha=\alpha+\pi$ and $\theta_\alpha=\alpha$.

For $\theta_\alpha=\alpha+\pi$, the second variation Γ_θ is negative definite. Hence θ is a local maximum of the potential H, and is therefore unstable in all directions. This state is classified as $(\infty, 0)$ in Fig. 5.

The state $\theta_\alpha=\alpha$ is locally stable for $K<2$, neutrally stable in two directions at the bifurcation point $K=2$, and unstable in two directions for $K>2$. As shown in Fig. 5, these stability types are denoted $(0,0)$, $(0,2)$ and $(2,0)$, respectively.

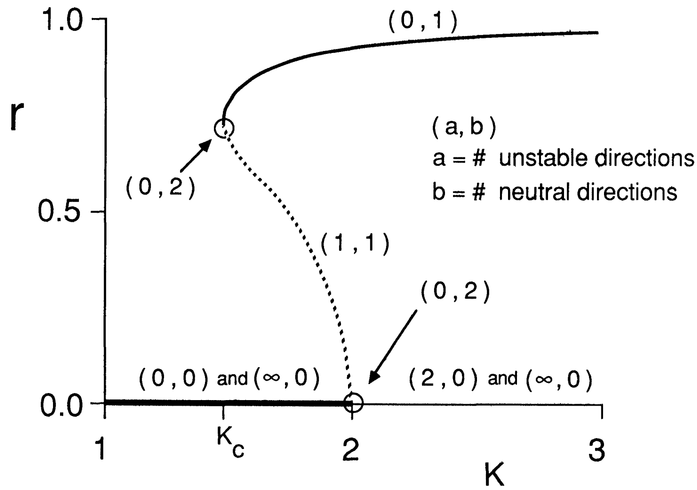

FIG. 5. *Stability diagram for the critical states of* (1.5). *Compare Fig.* 2. *The notation* (a, b) *gives the number of negative and zero eigenvalues of the second variation* Γ_θ *at the critical states. Heavy lines, locally stable states; thin or broken lines, unstable states; circles, bifurcation points. At* $r=0$ *there are two different types of critical states (see Case* 1 *in* § 3.4).

Case 2. $r>0$ (coherent states). The second variation Γ_θ given by (3.19) is positive definite on a space of codimension 2. Thus the critical states are stable in all but possibly two directions.

In one of the remaining directions, Γ_θ always has a zero eigenvalue (3.29). The existence of this neutrally stable direction stems from the invariance of the potential H under the group action (2.4). This group action is equivalent to a rotation of all the α's and θ's by the same amount (which changes ψ but leaves the potential invariant). The null direction is parallel to μ defined by (3.16), as shown in (3.31).

The remaining direction is parallel to ν. The corresponding eigenvalue determines the stability of the critical state $e^{i\theta_\alpha}=(u+e^{i\alpha})/|u+e^{i\alpha}|$. If $u>u_c$, the eigenvalue (3.22) for this direction is positive, indicating that the critical state is locally stable. At the bifurcation point $K=K_c$, the eigenvalue vanishes. The eigenvalue is negative for $u<u_c$ and the critical state is unstable. Hence as u crosses u_c from above, the stability type (Fig. 5) changes from (0, 1) to (1, 1). At the bifurcation the type is (0, 2).

4. Discussion. We have obtained exact solutions for the critical states of the spin system (1.5) and analyzed them for stability. The stability of these states was determined from the second variation of the system's potential function. Our main result is that the spin system (1.5) exhibits jump bifurcations and hysteresis as the coupling strength K is varied (Fig. 2). At small K, the system is incoherent ($r=0$) and the spins point in the random directions dictated by the local random magnetic field. As K is increased, this incoherent state persists until $K=2$, after which the spin system jumps discontinuously into a coherent configuration ($r\approx 1$) with all the spins nearly parallel. If K is now decreased below $K=2$, the system does not retrace its steps. It maintains coherence until $K=K_c\approx 1.489$, after which the system drops back into the incoherent state.

In the language of statistical physics, the system (1.5) undergoes a "first-order phase transition" [1]. Such discontinuous transitions have received less attention than the continuous "second-order phase transitions" [17], for which the apparatus of the renormalization group is well developed. However, first-order transitions, such as the freezing of water into ice, are extremely important in nature. The system studied here provides an exactly solvable example of a first-order transition.

The system (1.1) is known in physics as the "mean-field theory for the random-field XY model" [3], [26]. Our results give an exact solution for the magnetization in this model, under the assumptions of zero temperature and infinite-range coupling. The assumption of infinite-range coupling is admittedly unrealistic, but it is a useful and conventional starting point for more physically relevant calculations on short-range models.

Several authors have considered systems closely related to that studied here. Shinomoto and Kuramoto [20] presented numerical results about phase transitions in an XY model with external drive. They assumed a uniaxial magnetic field instead of the random field considered here. Villain and Fernandez [26] analyzed a random-field system similar to (1.1). However, they replaced the term $\sin(\theta_i - \theta_j)$ in (1.1) with its linearization $\theta_i - \theta_j$, in which case the model does not exhibit first-order transitions. Fisher [8], [9] and others [24] have studied a system similar to (1.1) as a model of change-density wave transport [10]. These authors also assumed a linear coupling term $\theta_i - \theta_j$ instead of the periodic coupling $\sin(\theta_i - \theta_j)$ in (1.1), and they too found only continuous transitions between incoherent and coherent states. However, hysteretic and discontinuous switching phenomena have been observed in experiments on change-density wave systems with strong pinning [12], [29]. We have shown recently [23] that a driven version of (1.1) accounts for these hysteretic phenomena in a physically reasonable way.

Appendix A. Discontinuous states. We may enlarge the set of admissible states to the space of all measurable maps $[0, 2\pi] \to [0, 2\pi]$, $\alpha \mapsto \theta_\alpha$. Such maps are, of course, Lebesgue integrable, and the potential

$$H = -\left[\frac{1}{2\pi}\int_0^{2\pi} \cos(\alpha - \theta_\alpha)\, d\alpha + \frac{K}{2}\cdot\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \sin(\theta_\beta - \theta_\alpha)\, d\alpha\, d\beta\right] \tag{A1}$$

is well-defined. Furthermore, the condition for a state θ_α to be critical is still

$$0 = \sin(\alpha - \theta_\alpha) + \frac{K}{2\pi}\int_0^{2\pi} \sin(\theta_\beta - \theta_\alpha)\, d\beta \tag{A2}$$

except with equality now required almost everywhere (a.e.). We define r, ψ as in (1.7) and assume without loss of generality that $\psi = 0$. Then

$$\sin(\alpha - \theta_\alpha) = rK \sin\theta_\alpha \text{ a.e.} \tag{A3}$$

is the condition for θ_α to be critical. Let $u = rK$.

Case 1 ($u = 0$). We solve $\sin(\alpha - \theta_\alpha) = 0$ a.e. This means $\theta_\alpha = \alpha$ or $\alpha + \pi$ a.e.; hence

$$\begin{aligned} \theta_\alpha &= \alpha & \alpha &\in A, \\ \theta_\alpha &= \alpha + \pi & \alpha &\in B, \end{aligned} \tag{A4}$$

for sets A, B where $A \cap B$ and $[0, 2\pi] \sim (A \cup B)$ have measure 0.

Then for any square integrable ϕ,

$$\Gamma_\theta(\phi) = \frac{1}{2\pi}\int_A \phi_\alpha^2\, d\alpha - \frac{1}{2\pi}\int_B \phi_\alpha^2\, d\alpha - K\left|\int_A \phi_\alpha\, d\alpha - \int_B \phi_\alpha\, d\alpha\right|. \tag{A5}$$

This form has infinitely many positive and negative eigenvalues, unless A or B has measure zero, and then θ is essentially continuous ($\theta_\alpha = \alpha$ or $\theta_\alpha = \alpha + \pi$, a.e.).

Case 2 ($u>0$, $u\neq 1$). Then

$$e^{i\theta_\alpha}=\pm\frac{u+e^{i\alpha}}{|u+e^{i\alpha}|},$$

so

$$\text{(A6)}\qquad \begin{aligned} e^{i\theta_\alpha}&=\frac{u+e^{i\alpha}}{|u+e^{i\alpha}|}, &\quad \alpha\in A,\\ e^{i\theta_\alpha}&=-\frac{u+e^{i\alpha}}{|u+e^{i\alpha}|}, &\quad \alpha\in B,\end{aligned}$$

A, B as above *and*

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i\theta_\alpha}=r>0.$$

Then

$$\text{(A7)}\qquad \cos(\theta_\alpha-\alpha)+u\cos\theta_\alpha=\begin{cases} |u+e^{i\alpha}| & \alpha\in A,\\ -|u+e^{i\alpha}| & \alpha\in B.\end{cases}$$

Hence for any ϕ

$$\text{(A8)}\qquad \begin{aligned}\Gamma_\theta(\phi)=&\frac{1}{2\pi}\int_A \phi_\alpha^2|u+e^{i\alpha}|\,d\alpha-\frac{1}{2\pi}\int_B \phi_\alpha^2|u+e^{i\alpha}|\,d\alpha\\ &-K\left|\frac{1}{2\pi}\int_0^{2\pi} e^{i\theta_\alpha}\phi_\alpha\,d\alpha\right|^2.\end{aligned}$$

This form also has infinitely many positive and negative eigenvalues, unless A or B has measure zero. If B has measure zero, θ is essentially our continuous critical state (2.13); if A has measure zero,

$$e^{i\theta_\alpha}=-\frac{u+e^{i\alpha}}{|u+e^{i\alpha}|}\text{ a.e.}$$

and $(1/2\pi)\int_0^{2\pi} e^{i\theta_\alpha}\,d\alpha<0$ which contradicts the hypothesis that $r>0$. Hence A cannot have measure zero.

Case 3 ($u=1$). The analysis below applies to the case $u=1$. We have $e^{i\theta_\alpha}=\pm e^{i\alpha/2}$; in order for the form Γ_θ to have finitely many negative eigenvalues, we need

$$\text{(A9)}\qquad \begin{aligned}\theta_\alpha&=\frac{\alpha}{2}, &\quad 0\leqq\alpha\leqq\pi,\\ \theta_\alpha&=\frac{\alpha}{2}+\pi, &\quad \pi\leqq\alpha\leqq 2\pi\quad\text{a.e.},\end{aligned}$$

that is,

$$\theta_\alpha=\frac{\alpha}{2},\qquad -\pi\leqq\alpha\leqq\pi\quad\text{a.e.}$$

Then

$$\text{(A10)}\qquad \Gamma_\theta(\phi)=\frac{1}{2\pi}\int_0^{2\pi}\phi_\alpha^2\cdot 2\left|\cos\frac{\alpha}{2}\right|\,d\alpha-K\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\alpha/2}\phi_\alpha\,d\alpha\right|^2.$$

We can study this form exactly as we did the continuous case $u>0$. Since $u=1<u_c$, this form will have one zero eigenvalue and one negative eigenvalue.

Appendix B. Elliptic integral for $f(u)$. The properties of the function $f(u)$ were determined by elementary methods in § 2.3. In this appendix we show how to rewrite $f(u)$ in terms of complete elliptic integrals of the first and second kinds.

The function $f(u)$ is defined by (2.17):

$$f(u)=\frac{1}{2\pi}\int_0^{2\pi}(u+\cos\alpha)(1+u^2+2u\cos\alpha)^{-1/2}\,d\alpha. \tag{B1}$$

Let

$$2\beta=\alpha, \tag{B2}$$

$$p=\frac{2\sqrt{u}}{1+u}. \tag{B3}$$

Then

$$\begin{aligned}(1+u^2+2u\cos\alpha)^{1/2}&=[1+u^2+2u(1-2\sin^2\beta)]^{1/2}\\ &=[(1+u)^2-4u\sin^2\beta]^{1/2}\\ &=(1+u)(1-p^2\sin^2\beta)^{1/2}.\end{aligned} \tag{B4}$$

The other factor $u+\cos\alpha$ in the integrand of (B1) becomes

$$\begin{aligned}u+\cos\alpha&=u+\cos 2\beta\\ &=(u+1)-2\sin^2\beta.\end{aligned} \tag{B5}$$

Hence

$$f(u)=\frac{2}{\pi(1+u)}\left[(1+u)\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-p^2\sin^2\beta}}-2\int_0^{\pi/2}\frac{\sin^2\beta\,d\beta}{\sqrt{1-p^2\sin^2\beta}}\right]. \tag{B6}$$

Using the standard definitions of the elliptic integrals

$$K(p)=\int_0^{\pi/2}\frac{d\beta}{\sqrt{1-p^2\sin^2\beta}}, \tag{B7}$$

$$D(p)=\int_0^{\pi/2}\frac{\sin^2\beta\,d\beta}{\sqrt{1-p^2\sin^2\beta}}, \tag{B8}$$

$$E(p)=\int_0^{\pi/2}\sqrt{1-p^2\sin^2\beta}\,d\beta, \tag{B9}$$

(B6) becomes

$$f(u)=\frac{2}{\pi(1+u)}[(1+u)K(p)-2D(p)]. \tag{B10}$$

Using (B3) and the identity $D(p)=(K(p)-E(p))/p^2$, we can simplify (B10) to the final result:

$$f(u)=\frac{1}{\pi u}\left[(u-1)K\left(\frac{2\sqrt{u}}{1+u}\right)+(u+1)E\left(\frac{2\sqrt{u}}{u+1}\right)\right]. \tag{B11}$$

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