Chapter 4

Commuting

Every decade or so a new approach to teaching math comes along and creates fresh opportunities for parents to feel inadequate. Back in the 1960s, my parents were flabbergasted by their inability to help me with my second-grade homework. They'd never heard of base 3 or Venn diagrams.

Now the tables have turned. "Dad, can you show me how to do these multiplication problems?" *Sure,* I thought, until the headshaking began. "No, Dad, that's not how we're supposed to do it. That's the old-school method. Don't you know the lattice method? No? Well, what about partial products?"

These humbling sessions have prompted me to revisit multiplication from scratch.[i] And it's actually quite subtle, once you start to think about it.

Take the terminology. Does "seven times three" mean "seven added to itself three times"? Or "three added to itself seven times"?

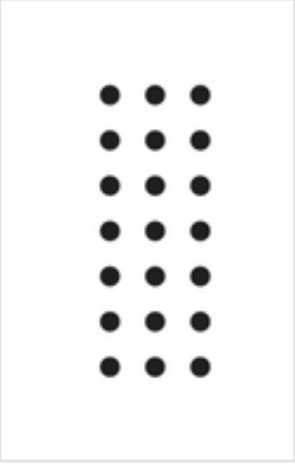
In some cultures the language is less ambiguous. A friend of mine from Belize used to recite his times tables like this: "Seven ones are seven, seven twos are fourteen, seven threes are twenty-one," and so on. This phrasing makes it clear that the first number is the multiplier; the second number is the thing being multiplied. It's the same convention as in Lionel Richie's immortal lyrics "She's once, twice, three times a lady." ("She's a lady times three" would never have been a hit.)

Maybe all this semantic fuss strikes you as silly, since the order in which numbers are multiplied doesn't matter anyway: 7 × 3 = 3 × 7. Fair enough, but that begs the question I'd like to explore in some depth here: Is this commutative law of multiplication, *a* × *b* = *b* × *a*, really so obvious? I remember being surprised by it as a child; maybe you were too.

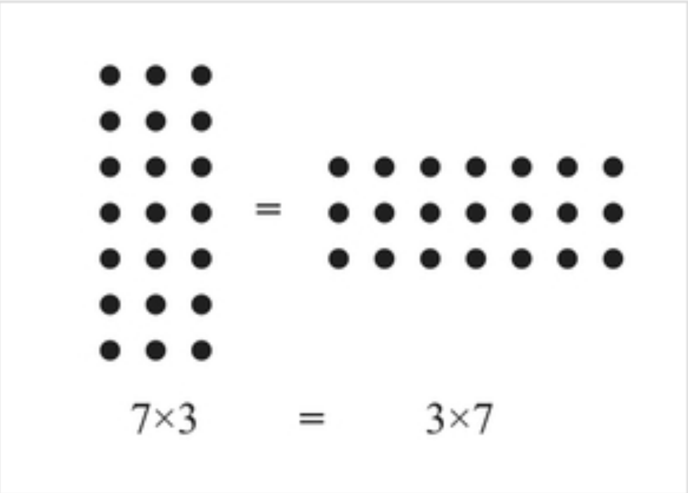
To recapture the magic, imagine not knowing what 7 × 3 equals. So you try counting by sevens: 7, 14, 21. Now turn it around and count by threes instead: 3, 6, 9, . . . Do you feel the suspense building? So far, none of the numbers match those in the sevens list, but keep going . . . 12, 15, 18, and then, bingo, 21!

My point is that if you regard multiplication as being synonymous with repeated counting by a certain number (or, in other words, with repeated addition), the commutative law isn't transparent.

But it becomes more intuitive if you conceive of multiplication *visually*. Think of 7 × 3 as the number of dots in a rectangular array with seven rows and three columns.



If you turn the array on its side, it transforms into three rows and seven columns—and since rotating the picture doesn't change the number of dots, it must be true that 7 × 3 = 3 × 7.



Yet strangely enough, in many real-world situations, especially where money is concerned, people seem to forget the commutative law, or don't realize it applies. Let me give you two examples.

Suppose you're shopping for a new pair of jeans.[ii] They're on sale for 20 percent off the sticker price of $50, which sounds like a bargain, but keep in mind that you also have to pay the 8 percent sales tax. After the clerk finishes complimenting you on the flattering fit, she starts ringing up the purchase but then pauses and whispers, in a conspiratorial tone, "Hey, let me save you some money. I'll apply the tax first, and then take twenty percent off the total, so you'll get more money back. Okay?"

But something about that sounds fishy to you. "No thanks," you say. "Could you please take the twenty percent off first, then apply the tax to the sale price? That way, I'll pay less tax."

Which way is a better deal for you? (Assume both are legal.)

When confronted with a question like this, many people approach it *additively*. They work out the tax and the discount under both scenarios, and then do whatever additions or subtractions are necessary to find the final price. Doing things the clerk's way, you reason, would cost you $4 in tax (8 percent of the sticker price of $50). That would bring your total to $54. Then applying the 20 percent discount to $54 gives you $10.80 back, so you'd end up paying $54 minus $10.80, which equals $43.20. Whereas under your scenario, the 20 percent discount would be applied first, saving you $10 off the $50 sticker price. Then the 8 percent tax on that reduced price of $40 would be $3.20, so you'd still end up paying $43.20. Amazing!

But it's merely the commutative law in action. To see why, think *multiplicatively,* not additively. Applying an 8 percent tax followed by a 20 percent discount amounts to multiplying the sticker price by 1.08 and then multiplying that result by 0.80. Switching the order of tax and discount reverses the multiplication, but since 1.08 × 0.80 = 0.80 × 1.08, the final price comes out the same.

Considerations like these also arise in larger financial decisions.[iii] Is a Roth 401(k) better or worse than a traditional retirement plan? More generally, if you have a pile of money to invest and you have to pay taxes on it at some point, is it better to take the tax bite at the beginning of the investment period, or at the end?

Once again, the commutative law shows it's a wash, all other things being equal (which, sadly, they often aren't). If, for both scenarios, your money grows by the same factor and gets taxed at the same rate, it doesn't matter whether you pay the taxes up front or at the end.

Please don't mistake this mathematical remark for financial advice. Anyone facing these decisions in real life needs to be aware of many complications that muddy the waters: Do you expect to be in a higher or lower tax bracket when you retire? Will you max out your contribution limits? Do you think the government will change its policies about the tax-exempt status of withdrawals by the time you're ready to take the money out? Leaving all this aside (and don't get me wrong, it's all important; I'm just trying to focus here on a simpler mathematical issue), my basic point is that the commutative law is relevant to the analysis of such decisions.

You can find heated debates about this on personal finance sites on the Internet. Even after the relevance of the commutative law has been pointed out, some bloggers don't accept it. It's that counterintuitive.

Maybe we're wired to doubt the commutative law because in daily life, it usually matters what you do first. You can't have your cake and eat it too. And when taking off your shoes and socks, you've got to get the sequencing right.

The physicist Murray Gell-Mann came to a similar realization one day when he was worrying about his future. As an undergraduate at Yale, he desperately wanted to stay in the Ivy League for graduate school. Unfortunately Princeton rejected his application. Harvard said yes but seemed to be dragging its feet about providing the financial support he needed. His best option, though he found it depressing, was MIT. In Gell-Mann's eyes, MIT was a grubby technological institute, beneath his rarefied taste. Nevertheless, he accepted the offer. Years later he would explain that he had contemplated suicide at the time but decided against it once he realized that attending MIT and killing himself didn't commute.[iv] He could always go to MIT and commit suicide later if he had to, but not the other way around.

Gell-Mann had probably been sensitized to the importance of non-commutativity. As a quantum physicist he would have been acutely aware that at the deepest level, nature disobeys the commutative law. And it's a good thing, too. For the failure of commutativity is what makes the world the way it is. It's why matter is solid, and why atoms don't implode.

Specifically, early in the development of quantum mechanics,[v] Werner Heisenberg and Paul Dirac had discovered that nature follows a curious kind of logic in which *p* × *q***<does not equal sign>** *q* × *p*, where *p* and *q* represent the momentum and position of a quantum particle. Without that breakdown of the commutative law, there would be no Heisenberg uncertainty principle, atoms would collapse, and nothing would exist.

That's why you'd better mind your *p*'s and *q*'s. And tell your kids to do the same.

ENDNOTES

[i] *revisit multiplication from scratch:* Keith Devlin has written a provocative series of essays about the nature of multiplication: what it is, what it is not, and why certain ways of thinking about it are more valuable and reliable than others. He argues in favor of thinking of multiplication as scaling, not repeated addition, and shows that the two concepts are very different in real-world settings where units are involved. See his January 2011 blog post "What exactly is multiplication?" at http://www.maa.org/devlin/devlin_01_11.html, as well as three earlier posts from 2008: "It ain't no repeated addition" (http://www.maa.org/devlin/devlin_06_08.html);
"It's still not repeated addition" (http://www.maa.org/devlin/devlin_0708_08.html); and "Multiplication and those pesky British spellings" (http://www.maa.org/devlin/devlin_09_08.html). These essays generated a lot of discussion in the blogosphere, especially among schoolteachers. If you're short on time, I'd recommend reading the one from 2011 first.

[ii] *shopping for a new pair of jeans:* For the jeans example, the order in which the tax and discount are applied may not matter to you—in both scenarios you end up paying $43.20—but it makes a big difference to the government and the store! In the clerk's scenario (where you pay tax based on the original price), you would pay $4 in tax; in your scenario, only $3.20. So how can the final price come out the same? It's because in the clerk's scenario the store gets to keep $39.20, whereas in yours it would keep $40. I'm not sure what the law requires, and it may vary from place to place, but the rational thing would be for the government to charge sales tax based on the actual payment the store receives. Only your scenario satisfies this criterion. For further discussion, see http://www.facebook.com/TeachersofMathematics/posts/166897663338316.

[iii] *financial decisions:* For heated online arguments about the relative merits of a Roth 401(k) versus a traditional one, and whether the commutative law has anything to do with these issues, see the Finance Buff, "Commutative law of multiplication" (http://thefinancebuff.com/commutative-law-of-multiplication.html), and the Simple Dollar, "The new Roth 401(k) versus the traditional 401(k): Which is the better route?" (http://www.thesimpledollar.com/2007/06/20/the-new-roth-401k-versus-the-traditional-401k-which-is-the-better-route/).

[iv] *attending MIT and killing himself didn't commute:* This story about Murray Gell-Mann is recounted in G. Johnson, *Strange Beauty* (Knopf, 1999), p. 55. In Gell-Mann's own words, he was offered admission to the "dreaded" Massachusetts Institute of Technology at the same time as he was "contemplating suicide, as befits someone rejected from the Ivy League. It occurred to me however (and it is an interesting example of non-commutation of operators) that I could try M.I.T. first and kill myself later, while the reverse order of events was impossible." This excerpt appears in H. Fritzsch, *Murray Gell-Mann: Selected Papers* (World Scientific, 2009), p. 298.

[v] *development of quantum mechanics:* For an account of how Heisenberg and Dirac discovered the role of non-commuting variables in quantum mechanics, see G. Farmelo, *The Strangest Man* (Basic Books, 2009), pp. 85–87.