Physica D 48 (1991) 102–112
North-Holland

DYNAMICS OF A GLOBALLY COUPLED OSCILLATOR ARRAY

Kwok Yeung TSANG[a,1], Renato E. MIROLLO[b], Steven H. STROGATZ[c] and Kurt WIESENFELD[a]
[a]*School of Physics, Georgia Tech, Atlanta, GA 30332, USA*
[b]*Department of Mathematics, Boston College, Chestnut Hill, MA 02167, USA*
[c]*Department of Mathematics, MIT, Cambridge, MA 02139, USA*

Received 16 March 1990
Revised manuscript received 12 September 1990
Accepted 12 September 1990
Communicated by J. Guckenheimer

We study a set of N globally coupled ordinary differential equations of the form encountered in circuit analysis of superconducting Josephson junction arrays. Particular attention is paid to two kinds of simple time-periodic behavior, known as in-phase and splay phase states. Some results valid for general N, as well as further results for $N=2$ and $N\to\infty$, are presented; a recurring theme is the appearance of very weak dynamics near the periodic states. The implications for Josephson junction arrays are discussed.

1. Introduction

The nonlinear dynamics of coupled oscillators has generated increasing attention over the last several years. Such systems arise naturally in the context of both biology [1–5] and physics [6–21]. The behavior of even a single nonlinear oscillator can be enormously complicated; the situation for N such elements is bound to be much worse. However, in many applications, the interest is in temporally simple behavior, with all elements oscillating with the same frequency (or nearly so). Often, the main question is whether the ensemble of oscillators will synchronize or not; more generally, one wishes to determine the conditions which enhance the tendency to synchronize.

In this paper, we study a particular set of ordinary differential equations, describing the dynamics of N identical oscillators. The motivation for studying these equations grew out of ongoing work on series arrays of Josephson junctions, which are superconducting electronic devices capable of generating extraordinarily high frequency voltage oscillations, up to 10^{11} Hz or more. In such devices, it is particularly desirable that the elements oscillate perfectly in phase, i.e. in perfect synchrony, in order that the power output reaches practically useful levels. Numerical studies of the Josephson arrays indicated that stable in-phase operation is possible; however, the degree of stability has been observed to depend greatly on the circuit geometry and the type of load used [10]. In order to get a better analytical understanding of the stability properties of Josephson arrays, we decided to write down a prototype model which retained the essential symmetry and coupling of the more complicated circuit equations:

$$\frac{\mathrm{d}\phi_k}{\mathrm{d}t} = \Omega + a\sin\phi_k + \frac{1}{N}\sum_{j=1}^{N}\sin\phi_j \tag{1}$$

for $k = 1, 2, \ldots, N$. One important feature of these

[1]Present address: Naval Research Laboratory, Code 4700.3, Washington, DC 20375, USA.

equations is the presence of "global coupling", wherein each oscillator is coupled with equal strength to all others. At first glance, this may seem to be an artificial coupling, but it arises quite naturally in (at least) two physical contexts. In fact, eq. (1) can be derived for a particular choice of Josephson junction circuit – as we show in section 2. Having said this, we emphasize that, for the purposes of the present paper, our interest is to gain as complete an understanding of eq. (1) as possible, including the limit of very large N.

The contents of this paper are as follows. In the next section, we give a general discussion of how globally coupled dynamics can arise in the description of physical systems, and then show in detail the relation of eq. (1) to a specific Josephson array circuit. We also define the in-phase and splay-phase periodic states observed in those circuit equations. In section 3, we present results valid for general N, concerning the stability properties of both in-phase and splay periodic orbits, which follow solely from the symmetries of the governing equations. Section 4 examines more specifically the case of $N = 2$: for some range of parameter values, the phase space divides into two parts, one part attracting to a sink, the other filled (foliated) with a continuous family of neutrally stable periodic orbits. This structure is due to a kind of time-reversibility of the dynamical system. In section 5, we analyze the splay states in the limit of $N \to \infty$; finally, we discuss some of the more intriguing aspects of the observed dynamics, as well as the relevance of these results for other oscillator arrays (in particular for Josephson junction arrays) in section 6.

2. Globally coupled arrays

In general, the basic structure of the governing equations plays an important part in the observed dynamics. This is especially true for the system studied in this paper. Eq. (1) possesses a high symmetry, owing to the fact that (i) the array consists of identical elements, and (ii) the coupling is global. This kind of coupling is most familiar to physicists within the context of "mean field theories" of statistical mechanics. In such mean field treatments, the coupling term represents a convenient approximation to the true underlying dynamics, which might, for example, take the form of a sum restricted to nearest neighbors. However, the appearance of global coupling does not always arise as the result of an approximation. Rather, it can happen that it correctly describes the relevant physical interactions between elements. This is the case in at least two examples we know of. The first case is that of multimode lasers in which the longitudinal modes are coupled via an intracavity nonlinear crystal [22–24]. Here, the individual degrees of freedom are the intensities of the different lasing modes I_k, and their coupling arises from the dissipation of energy caused by the crystal, which converts the lower-frequency photons ("red light") into higher-frequency photons ("green light"). This coupling is global, since the crystal interacts with the total amount of incident red light. Thus, each mode variable I_k obeys a rate equation which contains a loss term proportional to ΣI_k.

The second case where global coupling arises naturally is for certain array electrical circuits whose dynamics can be described using the elementary "lump circuit" laws of Kirchhoff. In particular, consider the circuit schematic of fig. 1, which shows an array of N identical elements in

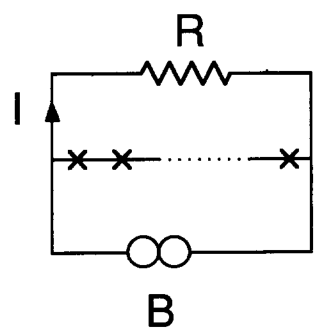

Fig. 1. Circuit schematic of a Josephson junction series array, subject to a parallel resistive load. Each cross represents a Josephson element.

parallel with a single resistor, and a constant current source [25, 26]. The source current B splits into two pieces, some flowing through the array I_k, and the rest through the load resistor I_L:

$$I_k + I_L = B, \tag{2}$$

a relation which is true for each element $k = 1, \ldots, N$. Meanwhile, the total voltage drop across the array – which is just the sum of the voltage drops across each element – is equal to the total voltage across the resistor:

$$\sum_{j=1}^{N} V_j = RI_L. \tag{3}$$

In general, the array elements are characterized by some current–voltage relation, say $V_k = f(I_k)$, where f may be a differential or integral operator. Together with (2) and (3) this yields:

$$I_k + \frac{1}{R} \sum_{j=1}^{N} f(I_j) = B.$$

One sees that the appearance of global coupling is the result of very simple considerations. (There are other circuit topologies which can lead to global coupling; naturally, there are circuit topologies that lead to different forms of coupling, as well.)

A single Josephson junction can be represented by two elements in parallel: an ideal junction which carries the supercurrent I_J, and a small resistor r which carries the normal current I_r. (For one class of Josephson junction – so-called tunnel junctions – one includes a capacitance as well.) Thus, $I_k = I_{Jk} + I_{rk}$ is the current conservation law for the kth element. The current–voltage relation for the ideal junction is usually represented via an intermediate variable ϕ. (ϕ represents the jump in the phase of the macroscopic quantum wavefunction across the junction gap.) Specifically, the supercurrent and voltage across the kth junction are given by

$$I_{Jk} = I_c \sin \phi_k, \tag{4}$$

$$V_k = \frac{\hbar}{2e} \frac{d\phi_k}{dt}, \tag{5}$$

$k = 1, \ldots, N$, where $\hbar$ is Planck's constant divided by 2π, e is the electron charge, and I_c is the critical current, a material-dependent parameter characterizing the Josephson junctions. Of course, the current I_{rk} across the junction resistor is just V_k/r, so that the left side of (2) becomes the sum of three terms:

$$I_c \sin \phi_k + V_k/r + I_L = B. \tag{6}$$

Combining (3), (5), and (6) yields

$$I_c \sin \phi_k + \frac{\hbar}{2er} \frac{d\phi_k}{dt} + \frac{\hbar}{2eR} \sum_j \frac{d\phi_j}{dt} = B. \tag{7}$$

If we sum (7) over all k, and use this result to eliminate the summation appearing in (7), we get

$$\frac{\hbar}{2e} \frac{R_0 + r}{I_c r^2} \frac{d\phi_k}{dt} = \frac{BR_0}{I_c r} - \frac{R_0 + r}{r} \sin \phi_k + \frac{1}{N} \sum_{j=1}^{N} \sin \phi_j, \tag{8}$$

where $R_0 = R/N$. A simple rescaling of time eliminates the coefficient on the left side. If we make the definitions: $\Omega = BR_0/I_c r$ and $a = -(R_0 + r)/r$, eq. (8) takes the form of eq. (1), as claimed. Note that $a < 0$ for ordinary resistors.

In the remainder of this paper, we will study the properties of eq. (1), without paying specific attention to its relationship with the circuit of fig. 1. Although many results will carry over, we note in particular that certain ranges of the a–Ω parameter plane correspond to negative resistance R. In this paper, we are primarily interested in eq. (1) as an example of a globally coupled oscillator array.

In both the study of Josephson junction arrays and the multimode laser problem, the governing array equations reveal solutions spanning the range from the very simple to the chaotic. For practical purposes, one is most interested in periodic solutions. In particular, two kinds of periodic solutions have received special attention. The first is the in-phase state, in which all the oscillators are perfectly synchronized and in lock step: $\phi_k(t) = \phi_0(t)$ for all k. The second type is what we shall call the "splay-phase" state, in which all oscillators have the same waveform, but are shifted by a fixed fraction of a wavelength: $\phi_k(t) = \phi_0(t + kT/N)$, where T is the oscillation period. (These "splay states" have been called "antiphase states" [25, 26] and "ponies on a merry-go-round" [27, 28] in other work on Josephson junction arrays; in numerical studies of the multimode laser equations, these are the "left waltz" and "right waltz" solutions described by James et al [23, 24]; in electrical analog simulations of van der Pol oscillators, these were called "rotating wave" solutions [29].) In applications, it is desirable to operate in the in-phase state; however, the splay states often stably coexist with the in-phase state, leading to a competition for available phase space [26, 30, 31].

3. Symmetries and results for general N

As a general rule, the underlying symmetries of a dynamical system have a profound effect on the observed dynamics [32, 33]. Eq. (1) has two important symmetries, which allow us to deduce some results valid for any N. The symmetries are:

(i) S_N *symmetry*. Eq. (1) is symmetric under any permutation of the N indices. This symmetry does not depend on the form of the terms in eq. (1); it obtains in any system of N identical oscillators which are globally coupled.

(ii) *Reversibility*. This symmetry *does* depend on the specific form of terms in eq. (1). In particular, if we define the shifted variables $\theta_k = \phi_k - \pi/2$, eq. (1) becomes

$$\frac{\mathrm{d}\theta_k}{\mathrm{d}t} = \Omega + a\cos\theta_k + \frac{1}{N}\sum_{j=1}^{N}\cos\theta_j \tag{9}$$

which is symmetric under $\theta_k \to -\theta_k, t \to -t$. Thus, the system is "reversible" in the following sense: if $\theta_k(t)$ is a solution, then so is $-\theta_k(-t)$.

These symmetries have some dramatic consequences for the resulting dynamics, as we now discuss.

Result 1. The in-phase periodic orbit is not asymptotically stable, for any N and any choice of parameter values. (Of course, this state exists only if there are no in-phase fixed points, which requires $|\Omega| > |a+1|$.)

This result follows from reversibility. To show this, suppose that there is an asymptotically stable in-phase periodic orbit Γ. Then, any initial condition in a tubular neighborhood of Γ must approach Γ as $t \to \infty$. However, for any such initial point $\{\theta_k\}$, the symmetry-related point $\{-\theta_k\}$ must also approach Γ, but as $t \to -\infty$. Thus, Γ is neither an attractor nor a repeller.

Note that this argument makes no use of the S_N symmetry. Therefore, a similar statement can be made for other array geometries: the in-phase state is not an attractor for any array whose governing equations have the reversibility symmetry. For example, it holds true for a ring of elements subject to nearest neighbor coupling. We will return to this point in section 6.

A linear stability analysis of the periodic in-phase state of eq. (1), based on a direct analysis of the equation, has shown that the state is linearly neutrally stable [11]. In fact, the linear part of the Poincaré map is the identity. In this sense, the system exhibits very weak dynamics near the in-phase orbit.

Actually, a more general statement can be made. Consider *any* system of the form

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = f(\theta_i) + \frac{K}{N}\sum_{j=1}^{N} f(\theta_j),$$

$$i = 1,\ldots,N, \tag{10}$$

where $f(\theta)$ is 2π-periodic, and suppose that $f(\theta) > 0$ for all θ so that (10) has a periodic in-phase orbit. (Note that eq. (9) is of this form, for $f(\theta) = A + B\cos\theta$, with the constants A and B chosen appropriately.) Then we have the following result.

Result 2. The periodic in-phase state is linearly neutrally stable for any system of the form (10). Moreover, the linearized Poincaré map about the in-phase orbit is the identity.

The important point is that this result holds *even if the system is not reversible.*

To prove result 2, we calculate the linearized Poincaré map. Let $\theta(t)$ denote the in-phase orbit, and consider a small perturbation $\theta_i(t) = \theta(t) + \xi_i(t)$. Then the linearized system is

$$\frac{\mathrm{d}\xi_i}{\mathrm{d}t} = f'(\theta(t))\left(\xi_i + \frac{K}{N}\sum_{j=1}^{N} f(\xi_j)\right),$$

where the prime denotes differentiation with respect to the argument. If we change variables to $\mu = N^{-1}\Sigma_{j=1}^{N}\xi_j$ and $\eta_i = \xi_i - \xi_{i+1}$, $i = 1,\ldots, N-1$, the linearized system completely decouples:

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = (1+K)f'(\theta(t))\,\mu,$$

$$\frac{\mathrm{d}\eta_i}{\mathrm{d}t} = f'(\theta(t))\,\eta_i.$$

This system can be integrated directly, and so there is no need for Floquet theory. Beginning with the equation for μ, we obtain

$$\frac{\mathrm{d}\mu}{\mu} = (1+K)f'(\theta(t))\,\mathrm{d}t$$

$$= (1+K)f'(\theta(t))\,\frac{\mathrm{d}t}{\mathrm{d}\theta}\mathrm{d}\theta$$

$$= \frac{f'(\theta)}{f(\theta)}\,\mathrm{d}\theta,$$

since $\mathrm{d}\theta/\mathrm{d}t = (1+K)f(\theta)$ for the in-phase orbit $\theta(t)$. Thus, integrating over one cycle of period T, we find $\ln[\mu(T)/\mu(0)] = \ln f(2\pi) - \ln f(0) = 0$, so that $\mu(T) = \mu(0)$. Similarly, $\eta_i(T) = \eta_i(0)$ for all i, and hence the linear part of the Poincaré map is the identity, as claimed.

Note that result 2 is complementary to result 1. It provides a more detailed description of the dynamics near the in-phase orbit, but unlike result 1, it does not rule out the possibility that higher-order terms could stabilize the in-phase orbit.

We turn now to a discussion of the stability of the splay-phase periodic orbits. We conjecture that the splay states are always neutrally stable. This conjecture is suggested by both numerical experiments (see section 6, below), and by recent analytical results obtained from averaging theory [34]. However, we have been unable to prove the neutral stability of the splay states in general.

There is one class of splay states for which we have a rigorous result: suppose $\theta_k(t)$ is a splay state generated by an *odd* function Θ. In other words, suppose there exists a T-periodic function such that $\theta_k(t) = \Theta(t + kT/N)$ for all t and k, and $\Theta(-t) = -\Theta(t)$. We call this an "odd splay state".

Result 3. If a splay state is odd, it is neither an attractor nor a repeller.

In fact, we suspect that all splay states are odd, but we have been unable to prove this. Our suspicions are based partly on numerical computations of splay states, in which we have always found Θ to be odd, and partly on results for the limit $N \to \infty$, for which Θ is demonstrably odd, as can be shown by the methods of section 5.

To prove result 3, we use the fact that for odd Θ, the reversibility symmetry permutes the splay states among themselves. In particular, we have $-\theta_k(-t) = \theta_{N-k}(t)$, because

$$\begin{aligned} -\theta_k(-t) &= -\Theta(-t + kT/N) \\ &= \Theta(t - kT/N) \\ &= \Theta(t + (N-k)T/N) \\ &= \theta_{N-k}(t). \end{aligned}$$

Now since the splay state $\theta_{N-k}(t)$ is related to $\theta_k(t)$ by a permutation, they have the same stability type. Hence $\theta_k(t)$ has the same stability type as its time-reversed cousin $\theta_k(-t)$. But time-reversal interchanges attractors and repellers; consequently, the splay state $\theta_k(t)$ is neither an attractor nor a repeller, as claimed.

4. $N = 2$

For $N = 2$, phase space is a 2-torus. In this case, we can determine completely the character of the phase portraits for any choice of parameters (Ω, a). There are fixed points given by $\sin \phi_k = -\Omega/(a+1)$ for both $k = 1, 2$; consequently, there are no fixed points when $|\Omega/(a+1)| > 1$, and four fixed points – a sink, a source, and two saddles – when $|\Omega/(a+1)| < 1$.

For $N = 2$, the S_N and reversibility symmetries allow us to make a surprising observation: *every orbit either approaches a fixed point as* $|t| \to \infty$, *or it is periodic*. We will prove this by explicitly constructing the periodic orbits. Consider an initial point $(\alpha, -\alpha)$, i.e. a point on the line $\theta_1 + \theta_2 = 0$ (see fig. 2). We integrate eq. (9) forward in time until the orbit intersects the line $\theta_1 + \theta_2 = 2\pi$, and call the point of intersection (β, γ), so that $\beta + \gamma = 2\pi$. Now consider the inversion-related initial point $(-\alpha, \alpha)$, and study the orbit flowing away from it both forward and backward in time. By the permutation symmetry, the

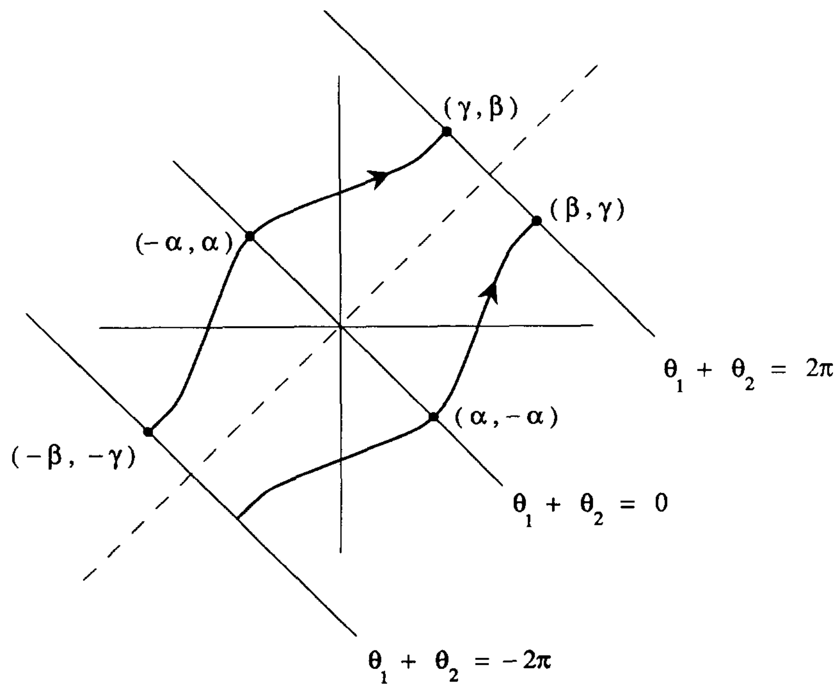

Fig. 2. Construction of periodic orbits for $N = 2$.

forward piece intersects the line $\theta_1 + \theta_2 = 2\pi$ at the point (γ, β); meanwhile, by the reversibility symmetry, the backward piece intersects the line $\theta_1 + \theta_2 = -2\pi$ at the point $(-\beta\ -\gamma)$. But these are the same endpoint, since $\beta = -\gamma\ (\mathrm{mod}\, 2\pi)$ – consequently, we have constructed a periodic orbit. This construction holds starting from any point on the line $\theta_1 + \theta_2 = 0$, provided it reaches the line $\theta_1 + \theta_2 = 2\pi$. Using index arguments [35], and the fact that the phase space is two-dimensional, one can show that all other orbits must approach a fixed point.

Thus far, we have made no explicit reference to the splay-phase state. In fact, when periodic

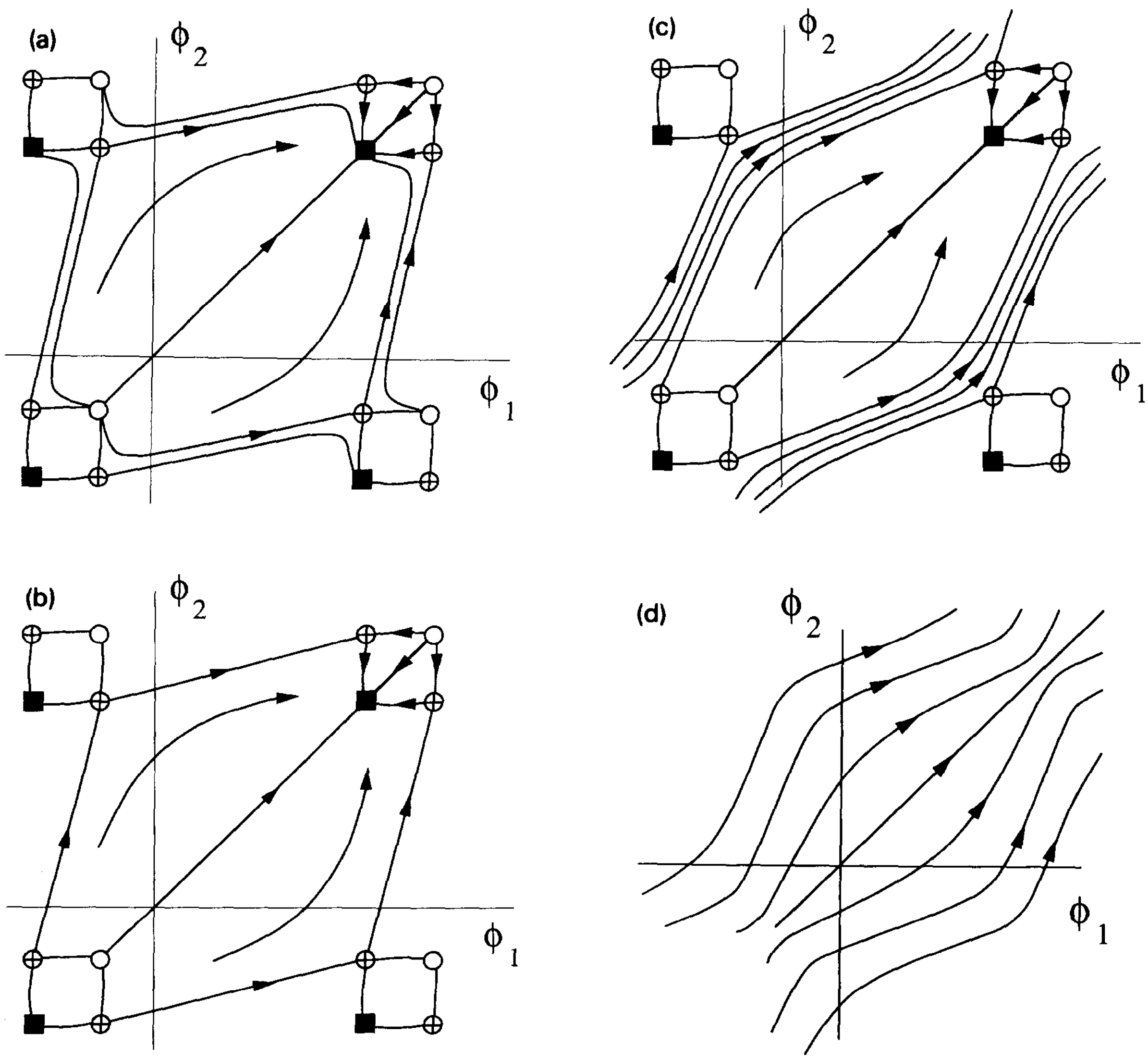

Fig. 3. Phase portraits for eq. (1) with $N = 2$ and $a > 0$. All of the pictures are periodic in both ϕ_1 and ϕ_2 directions. (■) sink, (○) source, (⊗) saddle. (a) For $\Omega < \Omega^*$, all orbits are attracted to the sink. (b) For $\Omega = \Omega^*$, the two saddles are connected by heteroclinic orbits. (c) For $\Omega^* < \Omega < a + 1$, two homoclinic orbits (each joining a saddle to itself) divide the torus into periodic and attracting regimes. Note the continuous band of periodic orbits. (d) For $a + 1 < \Omega$, all orbits are periodic and neutrally stable.

orbits exist at all, there is always a particular one which is a bona fide splay state. To see which one it is, one can think of the above construction as defining pairs of periodic orbits, one beginning from $(\alpha, -\alpha)$ and its inversion-related cousin beginning from $(-\alpha, \alpha)$, where α typically varies over an open set of values. However, there is a particular α for which the "pair" actually defines the same orbit. This is the one where $\alpha = -\beta = \gamma \pmod{2\pi}$; and this is precisely the splay-phase orbit.

We are now in a position to find the phase portraits for the system: these are shown in fig. 3a–3d, for various parameter ranges. The phase portraits shown are based on numerical integration for $a > 0$; they have not been proven rigorously. For $\Omega < \Omega^*$, all orbits are attracted to the sink (fig. 3a). At $\Omega = \Omega^*$, there is a bifurcation signalled by the appearance of two heteroclinic saddle connections (fig. 3b). Beyond this point, for $\Omega^* < \Omega < a + 1$, there is a homoclinic orbit for each saddle which serves to decompose the phase space into two regions (fig. 3c). In one region, all orbits are attracted to the sink; in the other region, one has a continuous family of (neutrally stable) periodic orbits. The periodic orbits do not all have the same period: the orbits lying closer to the homoclinic orbits have longer periods. As Ω increases further, this band of periodic orbits grows broader, until the four fixed points disappear (simultaneously) at $\Omega = a + 1$. At this point, the band has grown to be the whole torus, so that all points lie on neutrally stable periodic orbits (fig. 3d).

The flow depicted in fig. 3c is particularly curious, and deserves special attention: there is a region of dissipative behavior coexisting with a region of what one ordinarily associates with integrable behavior in Hamiltonian systems. In some sense, the system looks conservative or dissipative, depending on the initial conditions. This surprising situation persists over a range of parameter values. The origin of this phenomenon lies with the symmetries obeyed by our model [36]. We will return to this point in section 6.

5. Limit of large N

We turn next to the limit of very large arrays, $N \to \infty$, and consider the properties of the splay states. In this state, all the oscillators have the same waveform, but are shifted by a fixed fraction of a cycle: $\phi_k(t) = \Phi(t + kT/N)$ for $k = 1, \ldots, N$. Here, Φ is the common waveform and T is the oscillation period. Eq. (1) may be rewritten in terms of Φ as follows:

$$\frac{\mathrm{d}\phi_k}{\mathrm{d}t} = \Omega + a \sin \phi_k + \frac{1}{N} \sum_{j=1}^{N} \sin \Phi(t + jT/N).$$

As $N \to \infty$, we interpret the sum as a Riemann integral, with $\Delta t = T/N$. Then the sum converges to the time average of $\sin \Phi$,

$$\begin{aligned} &\frac{1}{N} \sum_{j=1}^{N} \sin \Phi(t + jT/N) \\ &\quad = \frac{1}{T} \sum_{j=1}^{N} \sin \Phi(t + jT/N)\, \Delta t \\ &\quad \to \frac{1}{T} \int_0^T \sin \Phi(t)\, \mathrm{d}t. \end{aligned}$$

Thus, $\Phi(t)$ satisfies the equation

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \Omega + a \sin \Phi + \frac{1}{T} \int_0^T \sin \Phi(t)\, \mathrm{d}t, \tag{11}$$

which may be solved self-consistently, as follows. Let

$$m = \frac{1}{T} \int_0^T \sin \Phi(t)\, \mathrm{d}t. \tag{12}$$

Then eq. (11) becomes simply

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \Omega + a \sin \Phi + m,$$

which has a period given by

$$T = \frac{2\pi}{\sqrt{(\Omega + m)^2 - a^2}}.$$

The quantity m can be determined from eq. (12) by changing variables from t to Φ, with the result

$$m = \frac{\sqrt{(\Omega+m)^2 - a^2}}{2\pi} \int_0^{2\pi} \frac{\sin\Phi \, d\Phi}{\Omega + m + a\sin\Phi}.$$

After evaluating the integral and solving for m, we find

$$m = \frac{-\Omega \pm \sqrt{\Omega^2 - a(a+2)}}{a+2}. \tag{13}$$

Thus, there are *two types* of splay states in the large-N limit, which we will call "plus" and "minus" corresponding to the choice of signs in eq. (13). These states are born together in a saddle–node bifurcation of periodic orbits [37] at $\Omega = \Omega^* = \sqrt{a(a+2)}$, and then move apart as Ω increases. The plus state continues to exist for all $\Omega > \Omega^*$. In contrast, the minus state cannot exist for arbitrarily large Ω, since m is the time average of $\sin\Phi$ and consequently bounded in magnitude. The critical case occurs when $\Omega = a + 1$ (so that $m = -1$ in eq. (13)). Remarkably, at this value of Ω, the minus splay state coalesces with the in-phase critical point $\phi_k = -\pi/2$. For larger values of Ω, there are no critical points at all, leaving the plus splay states coexisting with an in-phase periodic orbit.

One final remark: We find that the above value for Ω^* gives a surprisingly good estimate for the corresponding bifurcation point in the $N = 2$ case. For example, when $a = 1$, the above formula yields $\Omega^* = \sqrt{3}$ for $N \to \infty$, compared to our numerical result of $\Omega^* = 1.66 \pm .01$ for $N = 2$ (corresponding to fig. 3b).

6. Discussion

In this paper, we have focused on periodic solutions of the array eq. (1). Based on symmetry considerations alone, it is possible to draw conclusions for general N, regarding the lack of asymptotic stability of both in-phase and splay periodic orbits, when they exist. As long as there are no fixed points (that is, for $|\Omega/(a+1)| > 1$), it is easy to see that an in-phase periodic orbit exists; on the other hand, the existence of splay states has been established by direct analysis of the dynamical equations for $N = 2$ and $N \to \infty$. In fact, for $N = 2$ we have found a continuous band of periodic orbits; only one particular orbit in this band is a "true" splay state, as defined at the end of section 2. By topological [38] or degree-theoretic [28] arguments, one can prove the existence of bona fide splay states for any N, assuming Ω is sufficiently large.

We have found that the case of $N = 2$ is special, and return in particular to the situation depicted in fig. 3c. As we have noted, this phase portrait displays a coexistence of dissipative behavior (attraction to a sink) and "conservative" behavior – we use quotation marks because, for example, the flow is not area-preserving. Based on experience with Hamiltonian systems, we would associate the band of periodic orbits with some constant of motion like an "energy"; however, we have been unable to construct explicitly any such conserved quantity. This sort of dissipative/integrable coexistence has been reported previously, by Politi et al. [39], for equations governing a (somewhat idealized) laser system. In that work, the authors reported the "coexistence of conservative and dissipative dynamics" in a three-dimensional phase space. More recently, in a model of particle sedimentation in a highly viscous fluid, investigators also find conservative "Hamiltonian-like" behavior despite the presence of dissipation [36, 40]. As emphasized by Golubitsky et al., the origin of this behavior is the existence of "a time-reversal symmetry, which turns out to be just as good as a Hamiltonian structure for finding periodic solutions" [36]. It is interesting to note that, in the sedimentation problem, this occurs in the limit of zero inertia – similarly, the derivation of our model equations corresponds to the limit of vanishing capacitance of the Josephson junction elements (a limit valid for so-called point contact junctions, as opposed to tunnel junctions).

Next, we return to the relevance of our results for physically realizable Josephson junction arrays. First of all, the phase portrait depicted in fig. 3c corresponds to a negative load resistance (see fig. 1), and is therefore irrelevant under ordinary circumstances (i.e. using ordinary passive circuit elements). However, the results of section 3 are valid for any choice of parameters, and consequently hold for real arrays as well. In fact, since the arguments of that section depend only on symmetry considerations, the results may be generalized to other array geometries, for instance the two-dimensional arrays (which are of interest for their application as a new voltage standard), or other kinds of load circuits. And as already mentioned, neutral stability of the in-phase state holds for any array with "homogeneous" coupling, e.g. nearest neighbor coupling on a ring. Physically, the lack of asymptotic stability implies a sensitivity to the presence of external noise, such as Johnson noise generated in the circuit resistance. This sensitivity should be directly observable in experiments as an increased line width in the peaks of a power spectrum of the voltage output of the array, together with a noise rise at very low frequencies. A more quantitative analysis of these phenomena is currently underway [41].

We end by describing some intriguing empirical observations, based on numerical analysis. Recall that, for $N=2$, $a>0$, and $\Omega > a+1$, all orbits are periodic. What is the analogous statement for $N>2$? Computer simulations suggest a result that we find extremely surprising: for every N, the phase space appears to be foliated by a nested family of invariant 2-tori! It is conceivable that the system would have invariant 2-tori for $N=3$; what is amazing is that it appears to be true for N as large as we have investigated (up to $N=10$). These tori are concentric tubes about the splay states (essentially "inflated versions" of the splay state). When the system is started from a random initial condition, the trajectory runs "parallel" to the splay state and winds slowly about it. The trajectory is neither attracted to nor repelled from the splay state; instead it appears to be confined to the surface of a 2-torus (seen as an invariant closed curve in the Poincaré section). As time evolves, the trajectory simply runs around the torus in an apparently quasiperiodic fashion. Further investigation of these issues is in progress.

Motivated by an earlier version of this paper, Swift has devised an averaging method for studying the dynamics of eq. (9). The averaging theory predicts the observation of invariant 2-tori for all N, and also correctly accounts for the numerically observed ratios of the two frequencies [34]. Nevertheless, whether or not invariant 2-tori exist for the unaveraged system has yet to be determined rigorously.

Acknowledgements

We thank Neal Abraham, Raj Roy, and Nick Tufillaro for pointing our references to the laser literature, and David Schaeffer for suggesting the relevance of the references on particle sedimentation. We thank Don Aronson, Marty Golubitsky and Martin Krupa for valuable discussions on globally coupled arrays, and for supplying preprints of their work prior to publication. Finally, we thank Geza Gyorgyi, Charlie Marcus, and Jim Swift for valuable discussions. R.E.M. acknowledges support from the National Science Foundation under contract number DMS-8906423. S.H.S. acknowledges support from the National Science Foundation under contract number DMS-8916267, and a Presidential Young Investigator award. K.Y.T. and K.W. acknowledge support from the Office of Naval Research under contract number N00014-88-K-0494.

References

[1] A.T. Winfree, J. Theor. Biol. 16 (1967) 15.

[2] A.T. Winfree, The Geometry of Biological Time (Springer Berlin, 1980).

[3] N. Kopell and G.B. Ermentrout, Commun. Pure Appl. Math. 39 (1986) 623.
[4] L. Glass and M.C. Mackey, From Clocks to Chaos: The Rhythms of Life (Princeton Univ. Press, Princeton, 1988).
[5] N. Kopell, in: Neural Control of Rhythmic Movement in Vertebrates, eds. A.H. Cohen, S. Rossignol and S. Grillner (Wiley, New York, 1988).
[6] Y. Yamaguchi and H. Shimizu, Physica D 11 (1984) 212.
[7] Y. Kuramoto and I. Nishikawa, J. Stat. Phys. 49 (1987) 569.
[8] H. Sakaguchi, S. Shinomoto, and Y. Kuramoto, Prog. Theor. Phys. 77 (1987) 1005.
[9] H. Daido, Phys. Rev. Lett. 61 (1988) 231.
[10] P. Hadley, M.R. Beasley and K. Wiesenfeld, Phys. Rev. B 38 (1988) 8712.
[11] P. Hadley, M.R. Beasley and K. Wiesenfeld, Appl. Phys. Lett. 52 (1988) 1619.
[12] H. Sakaguchi, S. Shinomoto and Y. Kuramoto, Prog. Theor. Phys. 79 (1988) 1069.
[13] S.H. Strogatz and R.E. Mirollo, Physica D 31 (1988) 143.
[14] S.H. Strogatz and R.E. Mirollo, J. Phys. A 21 (1988) L699.
[15] S.H. Strogatz, C.M. Marcus, R.M. Westervelt and R.E. Mirollo, Phys. Rev. Lett. 61 (1988) 2380.
[16] S.S. Wang and H.G. Winful, Appl. Phys. Lett 52 (1988) 1774.
[17] S.S. Wang and H.G. Winful, Appl. Phys. Lett 53 (1988) 1894.
[18] J. Benford, H. Sze, W. Woo, R.R. Smith and B. Harteneck, Phys. Rev. Lett. 62 (1989) 969.
[19] M. Shiino and M. Frankowicz, Phys. Lett. A 136 (1989) 103.
[20] S.H. Strogatz, C.M. Marcus, R.M. Westervelt and R.E. Mirollo, Physica D 36 (1989) 23.
[21] R.E. Mirollo and S.H. Strogatz, SIAM J. Appl. Math. 50 (1990) 108.
[22] T. Baer, J. Opt. Soc. Am. B 3 (1986) 1175.
[23] G. E. James, Ph.D. Dissertation, Georgia Institute of Technology (1990), unpublished.
[24] G. James, E. Harrell III and R. Roy, Phys. Rev. A, to appear.
[25] P. Hadley and M.R. Beasley, Appl. Phys. Lett. 50 (1987) 621.
[26] P. Hadley, Ph.D. Dissertation, Stanford University (1989), unpublished.
[27] D.G. Aronson, M. Golubitsky and M. Krupa, Coupled arrays of Josephson junctions and bifurcations of maps with S_N symmetry, preprint (1990).
[28] D.G. Aronson, M. Golubitsky and J. Mallet-Paret, Ponies on a merry-go-round in large arrays of Josephson junctions, preprint (1990).
[29] P. Ashwin, G.P. King and J.W. Swift, Three identical oscillators with symmetric coupling, preprint (1990).
[30] K. Wiesenfeld and P. Hadley, Phys. Rev. Lett. 62 (1989) 1335.
[31] K.Y. Tsang and K. Wiesenfeld, Appl. Phys. Lett. 56 (1990) 495.
[32] M. Golubitsky and D.G. Schaeffer, Singularities and Groups in Bifurcation Theory, Appl. Math. Sci. 51 (Springer, Berlin, 1985).
[33] R.L. Devaney, in: Dynamical Systems Approaches to Nonlinear Problems in Systems and Circuits, eds. F.M.A. Salam and M.L. Levi (SIAM, Philadelphia, 1988).
[34] J.W. Swift, S.H. Strogatz, and K. Wiesenfeld, Averaging of globally coupled oscillators, in preparation (1990).
[35] D.W. Jordan and P. Smith, Nonlinear Ordinary Differential Equations (Oxford Univ. Press, Oxford, 1977).
[36] M. Golubitsky, M. Krupa and C.C. Lim, Time-reversibility and particle sedimentation, IMA Preprint Series No. 564 (August, 1989).
[37] J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields (Springer, Berlin, 1983).
[38] R.E. Mirollo, Splay phase orbits for equivariant flows on tori, preprint (1990).
[39] A. Politi, G.L. Oppo and R. Badii, Phys. Rev. A 33 (1986) 4055.
[40] R.E. Caflisch, C. Lim, J.H.C. Luke and A.S. Sangani, Phys. Fluids 31 (1988) 3175.
[41] K.Y. Tsang, R.E. Mirollo, S.H. Strogatz and K. Wiesenfeld, Reversibility and noise sensitivity of Josephson arrays, preprint (1990).