From a Swinging Chandelier to Global Positioning Systems

Calculus has unraveled mysteries that puzzled scientists for centuries, and it has led to technologies they never would have imagined.

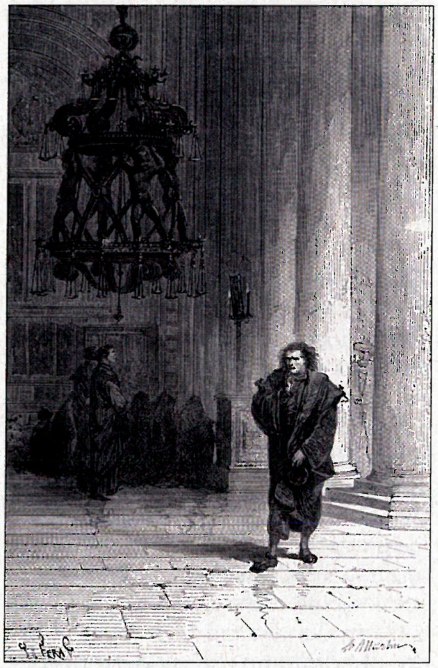
Steven Strogatz

Legend has it that Galileo Galilei (1564–1642) made his first scientific discovery when he was a teenage medical student. One day, while attending Mass at Pisa Cathedral, he noticed a chandelier swaying overhead, moving to and fro like a pendulum. Air currents kept jostling it, and Galileo observed that it always took the same time to complete its swing whether it traversed a wide arc or a small one. That surprised him. How could a big swing and a little swing take the same amount of time?

But the more he thought about it, the more it made sense. When the chandelier made a big swing, it traveled farther but it also moved faster. Maybe the two effects balanced out. To test this idea, Galileo timed the swinging chandelier with his pulse. Sure enough, every swing lasted the same number of heartbeats.

This legend is wonderful, and I want to believe it, but many historians doubt it happened. It comes down to us from Galileo's first and most devoted biographer, Vincenzo Viviani (1622–1703). As a young man, he had been Galileo's assistant and disciple near the end of the older man's life, when Galileo was completely blind and under house arrest. In his understandable reverence for his old master, Viviani was known to have embellished a tale or two when he wrote Galileo's biography years after his death.

But even if the story is apocryphal (and it may not be!), we do know for sure that Galileo performed careful experiments with pendulums as early as 1602 and that he wrote about them in 1638 in *Two New Sciences*. In that book, which is structured as a Socratic dialogue, one of the characters sounds like he was right there in the cathedral with the dreamy young student: "Thousands of times I have observed vibrations especially in churches where lamps, suspended by long cords, had been inadvertently set into motion." The rest of the dialogue expounds on the claim that a pendulum takes the same amount of time to traverse an arc of any size. So we know that Galileo was thoroughly familiar with the phenomenon described in Viviani's story; whether he actually discovered it as a teenager is anybody's guess.



World History Archive/Alamy Stock Photo

This 19th-century engraving depicts the legendary story of Galileo Galilei at the moment of inspiration as he watches the swaying chandelier at Pisa Cathedral. Physicists and mathematicians have built upon Galileo's early observations of pendulums and, with the application of calculus, have developed many of the technologies that shape the modern world.

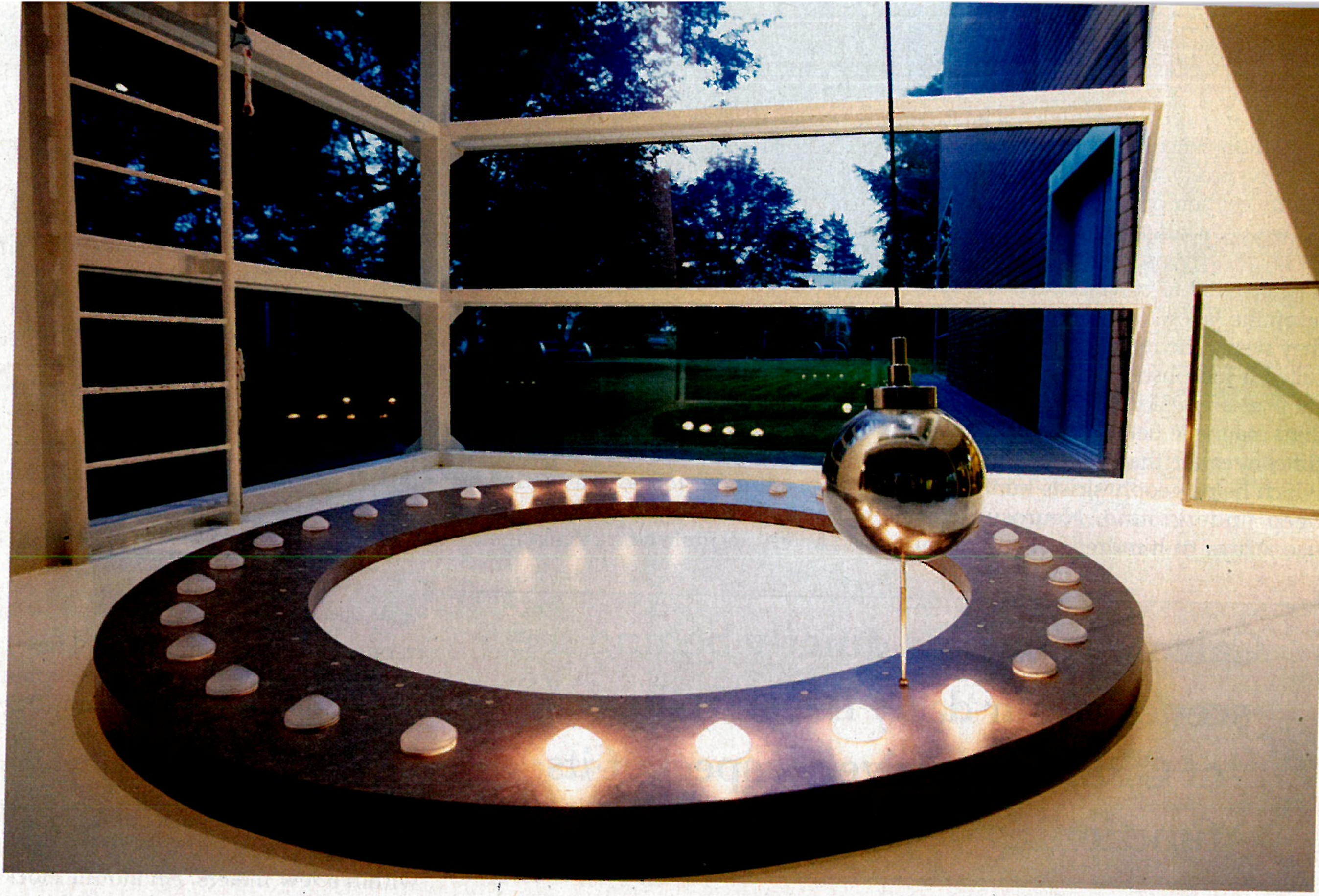
In any case, Galileo's assertion that a pendulum's swing always takes the same amount of time is not exactly true; bigger swings take a little longer. But if the arc is small enough—less than 20 degrees, say—it's very nearly true. This invariance of tempo for small swings is known today as the pendulum's *isochronism*, from the Greek words for "equal time." It forms the theoretical basis for metronomes and pendulum clocks, from ordinary grandfather clocks to the towering clock used in London's Big Ben. Galileo himself designed the world's first pendulum clock in the last year of his life, but he died before it could be built. The first working pendulum clock appeared 15 years later, invented by the Dutch mathematician and physicist Christiaan Huygens.

QUICK TAKE

Galileo Galilei studied pendulums in the 16th century, but their full potential would not be realized until the discovery of calculus a century later.

The sway of a pendulum regulates the timekeeping of metronomes and grandfather clocks, and the concept behind the movement extends to any vibrating object.

Oscillating ions operate on the same principle as pendulums. Their regularity provides atomic clocks with the precision required to operate the Global Positioning System.



Javier Larrea/agefotostock/Alamy Stock Photo

Foucault pendulums, such as this one at Eureka! Zientzia Museoa in San Sebastián, Spain, demonstrate the Earth's rotation. The pendulum swings along a consistent plane of oscillation while the planet spins below it, moving the room around the pendulum. Lights that turn on as the bob passes by show the pendulum's path. The consistency of a pendulum's path also applies to other oscillating objects, from generators to electrons.

Galileo was particularly intrigued—and frustrated—by a curious fact he discovered about pendulums: the elegant relationship between its length and its period (the time it takes the pendulum to swing once back and forth). As he explained, "If one wishes to make the vibration-time of one pendulum twice that of another, he must make its suspension four times as long." Using the language of proportions, he stated the general rule. "For bodies suspended by threads of different lengths," he wrote, "the lengths are to each other as the squares of the times." Unfortunately, Galileo never managed to derive this rule mathematically. It was an empirical pattern crying out for a theoretical explanation. He worked at it for years but failed to solve it. In retrospect, he couldn't have. Its explanation required a new kind of mathematics beyond any that he or his contemporaries knew. The derivation would have to wait until the late-17th century, for Isaac Newton and his discovery of the language of differential equations: calculus.

Paradigms of Oscillation

Galileo conceded that the study of pendulums "may appear to many exceedingly arid," although it was anything but that, as later work showed. In mathematics, pendulums stimulated the development of calculus through the riddles they posed. In physics and engineering, pendulums became paradigms of oscillation. Like the line in William Blake's poem, "Auguries of Innocence," about seeing a world in a grain of sand, physicists and engineers learned to see the world in a pendulum's swing. The same mathematics applied wherever oscillations occurred. The worrisome movements of a footbridge, the bouncing of a car with mushy shock absorbers, the thumping of a washing machine with an unbalanced load, the fluttering of venetian blinds in a gentle breeze, the rumbling of the Earth in the aftershock of an earthquake, the 60-cycle hum of fluorescent lights—every field of science and technology today has its own version of to-and-fro motion, of rhythmic return. The pendulum is the granddaddy of them all. Its patterns are universal. "Arid" is not the right word for them.

In some cases, the connections between pendulums and other phenomena are so exact that the same equations can be recycled without change. Only the symbols need to be reinterpreted; the syntax stays the same. It's as if nature keeps returning to the same motif again and again, a pendular repetition of a pendular theme. For example, the equations for the swinging of a pendulum carry over without change to those for the spinning of generators that produce alternating current and send it to our homes and offices. In honor of that pedigree, electrical engineers refer to their generator equations as *swing equations*.

The same equations pop up yet again, Zelig-like, in the quantum oscillations of a high-tech device that's billions of times faster and millions of times smaller than any generator or grandfather clock. In 1962 Brian Josephson, then a 22-year-old graduate student at the University of Cambridge, predicted that at temperatures close to absolute zero, pairs of superconducting electrons could tunnel back and forth through an impenetrable insulating barrier, a nonsensical statement according to classical physics. Yet calculus and quantum mechanics summoned these pendulum-like oscillations into existence—or, to put it less mystically, they revealed the possibility of their occurrence.

Two years after Josephson predicted these ghostly oscillations, the conditions needed to conjure them were set up in the laboratory and, indeed, there they were. The resulting device is now called a Josephson junction. Its practical uses are legion. It can detect ultrafaint magnetic fields a hundred billion times weaker than that of the Earth, which helps geophysicists hunt for oil deep underground. Neurosurgeons use arrays of hundreds of Josephson junctions to pinpoint the sites of brain tumors and locate the seizure-causing lesions in patients with epilepsy. The procedures are entirely noninvasive, unlike exploratory surgery. They work by mapping the subtle variations in magnetic field produced by abnormal electrical pathways in the brain. Josephson junctions could also provide the basis for extremely fast chips in the next generation of computers and might even play a role in quantum computation, which will revolutionize computer science if it ever comes to pass.

Calculus operates quietly behind the scenes of our daily lives. In the case of GPS, almost every aspect of its functioning depends on calculus.

Keeping Time

Pendulums also gave humanity the first way to keep time accurately. Until pendulum clocks came along, the best clocks were pitiful. They would lose or gain 15 minutes a day, even under ideal conditions. Pendulum clocks could be made a hundred times more accurate than that. They offered the first real hope of solving the greatest technological challenge of Galileo's era: finding a way to determine longitude at sea.

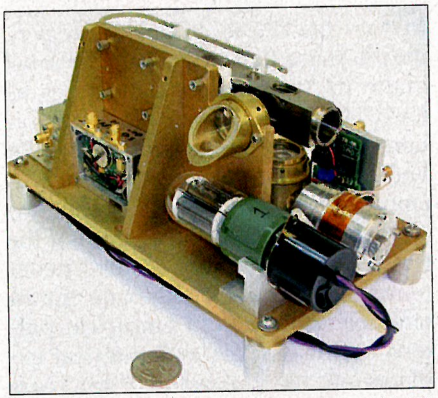
Unlike latitude, which can be ascertained by looking at the Sun or the stars, longitude has no counterpart in the physical environment. It is an artificial, arbitrary construct. But the problem of measuring it was real. In the age of exploration, sailors took to the oceans to wage war or conduct trade, but they often lost their way or ran aground because of confusion about where they were. The governments of Portugal, Spain, England, and Holland offered vast rewards to anyone who could solve the longitude problem. It was a challenge of the gravest concern.

When Galileo was trying to devise a pendulum clock in his last year of life, he had the longitude problem firmly in mind. He knew, as scientists had known since the 1500s, that the longitude problem could be solved if one had a very accurate clock. A navigator could set the clock at his port of departure and carry his home time out to sea. To determine the ship's longitude as it traveled east or west, the navigator could consult the clock at the exact moment of local noon, when the Sun was highest in the sky. Because the Earth spins through 360 degrees of longitude in a 24-hour day, each hour of discrepancy between local time and home time corresponds to 15 degrees of longitude. In terms of distance, 15 degrees translates to a whopping 1,600 kilometers at the equator. So for this scheme to have any hope of guiding a ship to its desired destination, give or take a few kilometers of tolerable error, a clock had to run true to within a few seconds a day. And it had to maintain this unwavering accuracy in the face of heaving seas and violent fluctuations in air pressure, temperature, salinity, and humidity, factors that could rust a clock's gears, stretch its springs, or thicken its lubricants, causing it to speed up, slow down, or stop.

Galileo died before he could build his clock and use it to tackle the longitude problem. Huygens presented his pendulum clocks to the Royal Society of London as a possible solution, but they were judged unsatisfactory because they were too sensitive to disturbances in their environment. Huygens later invented a marine chronometer whose ticktock oscillations were regulated by a balance wheel and a spiral spring instead of a pendulum, an innovative design that paved the way for pocket watches and modern wristwatches. In the end, however, the longitude problem was solved by a new kind of clock that used a spring-based mechanism, which was developed in the mid-1700s by John Harrison, an Englishman with no formal education. When tested at sea in the 1760s, his H4 chronometer tracked longitude to an accuracy of 16 kilometers, sufficient to win the British Parliament's prize of £20,000 (equivalent to a few million dollars today).

Navigating via Pendulum

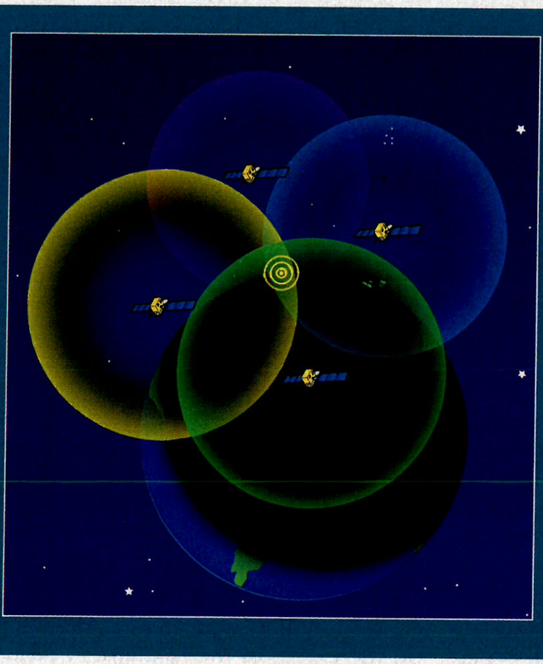
In our own era, the challenge of navigating on Earth still relies on the precise measurement of time. Consider the Global Positioning System (GPS). Just as mechanical clocks were the key to the longitude problem, atomic clocks are the key to pinpointing the location of anything on Earth to within a few meters. An atomic clock is a modern-day version of Galileo's pendulum clock. Like its forebear, it keeps time by counting oscillations, but instead of tracking the movements of a pendulum bob swinging back and forth, an atomic clock counts the vibrations of cesium atoms as they switch back and forth between two of their energy states, something they do when driven by microwaves that have a frequency of 9,192,631,770 cycles per second. Though the mecha-



NASA/JPL

NASA's Jet Propulsion Laboratory has developed the Deep Space Atomic Clock, which will allow spacecraft in deep space to navigate independently. The clock will keep time by counting the oscillations of mercury ions, applying the same principle used to measure the pendulum swing of a chandelier. These oscillations are so reliable that the clock only loses one second every 10 million years.

Never Get Lost Again


NASA/JPL

It can be difficult to remember a time before smartphones, when people relied on unwieldy paper maps to find their way through unfamiliar cities. Today, your phone uses the Global Positioning System (GPS) to pinpoint your location and determine the best route from here to there, faster than you can open your glove compartment.

A constellation of more than 30 GPS satellites orbits Earth, each carrying atomic clocks that count the pendulum-like oscillations of cesium or—in more advanced clocks—rubidium ions. A terrestrial network of antennae and monitor stations track the precise locations of the satellites, which are spread over six orbital planes—this configuration ensures that ground-based GPS devices can view at least four satellites from almost any location on the planet. Your phone calculates its distance from each satellite based on how long it takes signals to reach them. GPS then uses trilateration to identify your location with meter-level accuracy nearly anywhere on Earth.

nism is different, the principle is the same. Repetitive motion, back and forth, can be used to keep time.

And time, in turn, can determine your location. When you use the maps application in your phone or car, your device receives wireless signals from at least four GPS satellites that are orbiting about 20,000 kilometers overhead. Each satellite carries four atomic clocks that are synchronized to within a billionth of a second of one another. The various satellites visible to your receiver send it a continuous stream of signals, each of which is time-stamped to the nanosecond. That's where the atomic clocks come in. Their tremendous temporal precision gets converted into the tremendous spatial precision we've come to expect from GPS.

The calculation relies on trilateration, which works like this: When the signals from the four satellites arrive at the receiver, your GPS gadget compares the time they were received to the time they were transmitted. Those four times are all slightly different, because the satellites are at four different distances away from you. Your GPS device multiplies those four tiny time differences by the speed of light to calculate how far away you are from the four satellites overhead. Because the positions of the satellites are known and controlled extremely accurately, your GPS receiver can then calculate those four distances to determine where it is on the surface of the Earth. It can also figure out its elevation and speed. In essence, GPS converts very precise measurements of time into very precise measurements of distance and thereby into very precise measurements of location and motion.

GPS was developed by the U.S. military during the Cold War. The original intent was to keep track of U.S. submarines carrying nuclear missiles and give them precise estimates of their current locations so that if they needed to launch a nuclear strike, they could target their intercontinental ballistic missiles very accurately. Peacetime applications of GPS nowadays include precision farming, blind landings of airplanes in heavy fog, and enhanced 911 systems that automatically calculate the fastest routes for ambulances and fire trucks.

But GPS is more than a location and guidance system. It allows time synchronization to within 100 nanoseconds, which is useful for coordinating bank transfers and other financial transactions. It also keeps wireless phone and data networks in sync, allowing them to share the frequencies in the electromagnetic spectrum more efficiently.

I've gone into all this detail because GPS is a prime example of the hidden usefulness of calculus. As is so often the case, calculus operates quietly behind the scenes of our daily lives. In the case of GPS, almost every aspect of the functioning of the system depends on calculus. Think about the wireless communication between satellites and receivers; calculus predicted the electromagnetic waves that make the technology possible. Without calculus, there'd be no wireless and no GPS. Likewise, the atomic clocks on the GPS satellites use the quantum mechanical vibrations of cesium atoms; calculus underpins the equations of quantum mechanics and the methods for solving them.

I could go on—calculus underlies the mathematical methods for calculating the trajectories of the satellites and controlling their locations, and for incorporating Albert Einstein's relativistic corrections to the time measured by atomic clocks as they move at high speeds and in weak gravitational fields—but I hope the main point is clear. Calculus enabled the creation of much of what made GPS possible. Calculus didn't do it on its own, of course. It was a supporting player, but an important one. Along with electrical engineering, quantum physics, aerospace engineering, and all the rest, calculus was an indispensable part of the team.

So let's return to young Galileo sitting in the Pisa Cathedral pondering that chandelier swinging back and forth. We can see now that his idle thoughts about pendulums and the equal times of their swings had an outsize impact on the course of civilization, not just in his own era but in our own.

Steven Strogatz is the Jacob Gould Schurman Professor of Applied Mathematics at Cornell University. This article is adapted from his book Infinite Powers: How Calculus Reveals the Secrets of the Universe. *Copyright © 2019 by Steven Strogatz. Reprinted by permission of Houghton Mifflin Harcourt Publishing Company. All rights reserved. Website: www.stevenstrogatz.com*