

A global synchronization theorem for oscillators on a random graph

Cite as: Chaos 32, 093119 (2022); <https://doi.org/10.1063/5.0090443>

Submitted: 07 March 2022 • Accepted: 19 August 2022 • Published Online: 16 September 2022

Martin Kassabov,  Steven H. Strogatz and  Alex Townsend

COLLECTIONS

Paper published as part of the special topic on [Dynamics of Oscillator Populations](#)



[View Online](#)



[Export Citation](#)



[CrossMark](#)

APL Machine Learning

Open, quality research for the networking communities

Now Open for Submissions

[LEARN MORE](#)

AIP
Publishing

A global synchronization theorem for oscillators on a random graph

Cite as: Chaos 32, 093119 (2022); doi: 10.1063/5.0090443

Submitted: 7 March 2022 · Accepted: 19 August 2022 ·

Published Online: 16 September 2022



View Online



Export Citation



CrossMark

Martin Kassabov, Steven H. Strogatz,  and Alex Townsend^{a)} 

AFFILIATIONS

Department of Mathematics, Cornell University, Ithaca, New York 14853, USA

Note: This article is part of the Focus Issue, Dynamics of Oscillator Populations.

^{a)} **Author to whom correspondence should be addressed:** townsend@cornell.edu

ABSTRACT

Consider n identical Kuramoto oscillators on a random graph. Specifically, consider Erdős–Rényi random graphs in which any two oscillators are bidirectionally coupled with unit strength, independently and at random, with probability $0 \leq p \leq 1$. We say that a network is globally synchronizing if the oscillators converge to the all-in-phase synchronous state for almost all initial conditions. Is there a critical threshold for p above which global synchrony is extremely likely but below which it is extremely rare? It is suspected that a critical threshold exists and is close to the so-called connectivity threshold, namely, $p \sim \log(n)/n$ for $n \gg 1$. Ling, Xu, and Bandeira made the first progress toward proving a result in this direction: they showed that if $p \gg \log(n)/n^{1/3}$, then Erdős–Rényi networks of Kuramoto oscillators are globally synchronizing with high probability as $n \rightarrow \infty$. Here, we improve that result by showing that $p \gg \log^2(n)/n$ suffices. Our estimates are explicit: for example, we can say that there is more than a 99.9996% chance that a random network with $n = 10^6$ and $p > 0.01117$ is globally synchronizing.

Published under an exclusive license by AIP Publishing. <https://doi.org/10.1063/5.0090443>

Random graphs are fascinating topologies on which to study the dynamics of coupled oscillators. Despite the statistical nature of random graphs, coherent synchronization seems to be ubiquitous above a critical threshold. To investigate this, we consider the homogeneous version of the Kuramoto model in which all n oscillators have the same intrinsic frequency. For simplicity, the oscillators are arranged on an Erdős–Rényi random graph in which any two oscillators are coupled with unit strength by an undirected edge, independently at random, with some probability $0 \leq p \leq 1$; however, our proof strategy can be applied to other random graph models too. We say that a network is globally synchronizing if the oscillators converge to the all-in-phase synchronous state for almost all initial conditions. For what values of p is an Erdős–Rényi random network very likely to be globally synchronizing? Here, we prove that $p \gg \log^2(n)/n$ as $n \rightarrow \infty$ is a sufficient condition. Our proof uses trigonometric inequalities and an amplification argument involving the first two moments of the oscillator phase distribution that must hold for any stable phase-locked state. Specifically, we show that the spectral norms of the mean-centered adjacency and graph Laplacian matrix can be used to guarantee that a network is globally synchronizing. Our analysis is explicit, and we can reason about random networks of finite, practical size.

I. INTRODUCTION

Networks of coupled oscillators have long been studied in biology, physics, engineering, and nonlinear dynamics.^{1–13} Recently, they have begun to attract the attention of other communities as well. For example, oscillator networks have been recognized as having the potential to yield “beyond-Moore’s law” computational devices¹⁴ for graph coloring,¹⁵ image segmentation,¹⁶ and approximate maximum graph cuts.¹⁷ They have also become a model problem for understanding the global convergence of gradient descent in nonlinear optimization.¹⁸ In such settings, global issues come to the fore. When performing gradient descent, for instance, one typically wants to avoid getting stuck in local minima. Conditions to enforce convergence to the global minimum then become desirable. Likewise, conditions to enforce the global synchrony of oscillators play an analogous role in dynamical systems theory.

We say that a network of oscillators *globally synchronizes* if it converges to a state for which all the oscillators are in phase, starting from all initial conditions except a set of measure zero. Until recently, only a few global synchronization results were known for networks of oscillators.^{4,5} These results were restricted to complete graphs, in which each oscillator is coupled to all the others. In the past decade, however, several advances have been made for a wider class of network structures, starting with work by Taylor,¹⁹ who

proved that if each oscillator in a network of identical Kuramoto oscillators is connected to at least 94% of the others, the network will fall into perfect synchrony for almost all initial conditions, no matter what the topology of the network is like in other respects. Taylor’s result was strengthened by Ling *et al.*¹⁸ in 2018, and further progress has been made since then.^{20,21}

Ling *et al.*¹⁸ also made a seminal advance in the study of *random* networks. They considered identical Kuramoto oscillators on an Erdős–Rényi random graph,²² in which any two oscillators are coupled with unit strength, independently and at random, with probability $0 \leq p \leq 1$; otherwise, the oscillators are uncoupled. They showed that if $p \gg \log(n)/n^{1/3}$, then with high probability, the network is globally synchronizing as $n \rightarrow \infty$.

The open question is to find and prove the sharpest result along these lines. Intuitively, as one increases the value of p from 0 to 1, one expects to find a critical threshold above which global synchrony is extremely likely and below which it is extremely rare.

At the very least, for global synchrony to be ubiquitous, p must be large enough to ensure that the random network is connected, and from the theory of random graphs,²² we know that connectedness occurs with high probability once $p > (1 + \epsilon) \log(n)/n$ for any $\epsilon > 0$. Therefore, the critical threshold for global synchronization cannot be any smaller than this connectivity threshold and is apt to be a little above. On the basis of numerical evidence, Ling *et al.*¹⁸ conjectured that if $p \gg \log(n)/n$, then Erdős–Rényi graphs are globally synchronizing as $n \rightarrow \infty$, but nobody has proven that yet. The challenge is to see how close one can get. In this paper, we come within a factor of $\log(n)$ and prove that $p \gg \log^2(n)/n$ is sufficient to give global synchrony with high probability.

We study the homogeneous Kuramoto model^{19,23,24} in which each oscillator has the same frequency ω . By going into a rotating frame at this frequency, we can set $\omega = 0$ without loss of generality. Then, phase-locked states in the original frame correspond to equilibrium states in the rotating frame. Therefore, to explore the question that concerns us, it suffices to study the following simplified system of identical Kuramoto oscillators:

$$\frac{d\theta_j}{dt} = \sum_{k=1}^n A_{jk} \sin(\theta_k - \theta_j), \quad 1 \leq j \leq n, \quad (1)$$

where $\theta_j(t)$ is the phase of oscillator j (in the rotating frame) and the adjacency matrix A is randomly generated. In particular, $A_{jk} = A_{kj} = 1$ with probability p , with $A_{jk} = A_{kj} = 0$ otherwise, independently for $1 \leq j, k \leq n$. Thus, all interactions are assumed to be symmetric, equally attractive, and of unit strength. (The assumption that all interactions are equally attractive is made for convenience. We expect our arguments and results below can be generalized to attractive interactions of non-uniform strength.) We take the unusual convention that the network can have self-loops so that oscillator i is connected to itself with probability p ; i.e., $\mathbb{P}[A_{jj} = 1] = p$. Since $\sin(0) = 0$, this convention does not alter the dynamics of the oscillators, but it does make proof details easier to write down.

Finally, since the adjacency matrix A is symmetric, we know that (1) is a gradient system.^{23,24} Thus, all the attractors of (1) are equilibrium points. Therefore, we do not need to concern ourselves with the possibility of more complicated attractors, such as limit

cycles, tori, chimera states, or strange attractors. However, one slight complication is that the equilibria are not isolated; each equilibrium point lies on a subspace of equilibria because of the rotational symmetry of the Kuramoto model. Specifically, if $\theta = (\theta_1, \dots, \theta_n)$ is an equilibrium, then so is $\hat{\theta} = (\theta_1 + a, \dots, \theta_n + a)$ for any real number a . Despite this non-generic feature, the long-term dynamics are simple. Since the dynamics of the oscillators is governed by a gradient system and the gradient is zero along these subspaces of equilibria, each solution of (1) will still converge to an equilibrium point (that happens to lie on a subspace of equilibria). In fact, since the time derivative of $\sum_{j=1}^n \theta_j(t)$ is zero, the average of the θ_j ’s remains constant over time with the constant being determined by the initial conditions. This means that one can assume that the dynamics of the oscillators is restricted to a hyperplane on which there is a unique all-in-phase state. When we say “the all-in-phase state,” we mean the unique one in this invariant hyperplane.

We find it helpful to visualize the oscillators on the unit circle, where oscillator j is positioned at the coordinate $[\cos(\theta_j), \sin(\theta_j)]$. From this perspective, a network is globally synchronizing if starting from any initial positions (except a set of measure zero), all the oscillators eventually end up at the same point on the circle. Due to periodicity, we assume that the phases, θ_j , take values in the interval $[-\pi, \pi)$.

One cannot determine whether a network is globally synchronizing by numerical simulation of (1), as it is impossible to try all initial conditions. Of course, one can try millions of random initial conditions of the oscillators’ phases and then watch the dynamics of (1). However, even if all observed initial states eventually fall into the all-in-phase state, one cannot conclude that the system is globally synchronizing because other stable equilibria could still exist; their basins might be minuscule but could nevertheless have a positive measure. These small basins of attraction are very difficult to detect with random initial conditions.

With that caveat in mind, we note that such numerical experiments have been conducted, and they tentatively suggest that $p \gg \log(n)/n$ is sufficient for global synchronization.¹⁸ In this paper, we investigate global synchrony via a theoretical study. We show that $p \gg \log^2(n)/n$ is good enough to ensure global synchronization with high probability, improving on $p \gg \log(n)/n^{1/3}$ proved by Ling *et al.*,¹⁸ and bringing us closer to the connectivity threshold of Erdős–Rényi graphs.

Although we focus on Erdős–Rényi graphs, many of the inequalities we derive hold for any random or deterministic network. To highlight this, we state many of our findings for a general graph G and a general parameter $p \in \mathbb{R}$. In the end, we restrict ourselves back to Erdős–Rényi graphs and take p to be the probability of a connection between any two oscillators.

Our results depend on both the adjacency matrix A and the graph Laplacian matrix $L = D - A$, where D is a diagonal matrix and D_{ii} is the degree of vertex i (counting self-loops). For any $p \in \mathbb{R}$, denote the shifted adjacency and the graph Laplacian matrix by

$$\Delta_A = A - pJ_n, \quad \Delta_L = L + pJ_n - npI_n, \quad (2)$$

where J_n is the $n \times n$ matrix of all ones and I_n is the $n \times n$ identity matrix. It is worth noting that for Erdős–Rényi graphs, the shifts are precisely the expectation of the matrices as $\mathbb{E}[A] = pJ_n$

and $\mathbb{E}[L] = -pJ_n + npI_n$. Remarkably, we show that the global synchrony of a network can be guaranteed by ensuring that the spectral norms $\|\Delta_A\|$ and $\|\Delta_L\|$ satisfy particular inequalities. The spectral norm of a symmetric matrix is the maximum eigenvalue in absolute magnitude, and $\|\Delta_A\|$ and $\|\Delta_L\|$ are extensively studied in the random matrix literature.^{25,26} We also find it appealing that the spectral norm of the graph Laplacian matrix appears naturally in our analysis, as it has been used previously to study the dynamics of networks of oscillators²⁷ as well as diffusion on graphs. From a high-level viewpoint, we prove that global synchronization of a graph is a spectral property, like many other graph properties.

While we focus in this paper on global synchronization of identical oscillators, Medvedev and his collaborators have considered other aspects of synchronization for *non*-identical oscillators on Erdős–Rényi graphs, as well as on other graphs, such as Cayley and Watts–Strogatz graphs, often in the continuum limit.^{28–31} Their findings have a similar overarching message that synchronization occurs spontaneously above a critical threshold.

II. OVERVIEW OF OUR PROOF STRATEGY

To prove that a random graph is globally synchronizing with high probability, we bridge the gap between spectral graph theory and coupled oscillator theory. The literature contains many good probabilistic estimates for the spectral norm of a random graph’s adjacency and graph Laplacian matrices, which we use to control the long-time dynamics of (1). The key to our proof is to establish conditions on these two spectral norms that force any stable equilibrium to have phases that lie within a half-circle. Confining the phases in this way then guarantees global synchronization because it is known that the only stable equilibrium of (1) with phases confined to a half-circle is the all-in-phase synchronized state.^{8,23}

Our first attempt at controlling the phases is the inequality we state below as (9), which can be used to guarantee that very dense Erdős–Rényi graphs are globally synchronizing with high probability. As an aside, when $p = 1$ inequality (9) together with (4) provides a new proof that a complete graph of identical Kuramoto oscillators is globally synchronizing.⁵ A similar inequality to (9) was derived by Ling *et al.*¹⁸ to show that $p \gg \log(n)/n^{1/3}$ is sufficient for global synchrony with high probability.

To improve on (9), our argument becomes more intricate. We carefully examine the possible distribution of edges between oscillators whose phases lie on different arcs of the circle and show that any equilibrium is destabilized if there are too many edges between oscillators that have disparate phases (see Lemma 8 and Fig. 2).

III. BOUNDS ON THE ORDER PARAMETER AND ITS HIGHER-ORDER MOMENTS

An important quantity in the study of Kuramoto oscillators is the so-called complex order parameter, ρ_1 . The magnitude of ρ_1 is between 0 and 1 and measures the synchrony of the oscillators in the network. We find it useful to also look at second-order moments of the oscillator distribution for analyzing the synchrony of random networks. Higher-order moments are also called Daido order parameters and can be used to analyze oscillators with all-to-all coupling, corresponding to a complete graph.^{9,32–34}

For an equilibrium $\theta = (\theta_1, \dots, \theta_n)$, we define the first- and second-order moments as

$$\rho_1 = \frac{1}{n} \sum_j e^{i\theta_j}, \quad \rho_2 = \frac{1}{n} \sum_j e^{2i\theta_j}.$$

(For convenience, we use the notation \sum_j to mean $\sum_{j=1}^n$.) Without loss of generality, we may assume that the complex order parameter ρ_1 is real-valued and nonnegative. To see this, write $\rho_1 = |\rho_1|e^{i\psi}$ for some ψ . Then, $\hat{\theta} = (\theta_1 - \psi, \dots, \theta_n - \psi)$ is also an equilibrium of (1) with the same stability properties as θ since (1) is invariant under a global shift of all phases by ψ . Therefore, for the rest of this paper, we assume that $\psi = 0$ for any equilibrium of interest with $0 \leq \rho_1 \leq 1$. This allows us to select a representative equilibrium point from its associated subspace of equilibria. When $\rho_1 = 1$, the oscillators are in pure synchrony, and when $\rho_1 \approx 0$, the phases are scattered around the unit circle without a dominant phase.

To avoid working with complex numbers, it is convenient to consider the quantity $|\rho_2|^2$. For $m = 1, 2$, we have

$$|\rho_m|^2 = \frac{1}{n^2} \left(\sum_k e^{im\theta_k} \sum_j e^{-im\theta_j} \right) = \frac{1}{n^2} \sum_{j,k} \cos(m(\theta_k - \theta_j)). \quad (3)$$

By analyzing ρ_1^2 and $|\rho_2|^2$, one hopes to witness the rough statistics of an equilibrium to understand its potential for synchrony without concern for its precise pattern of phases.

Let $q_\theta = (e^{i\theta_1}, \dots, e^{i\theta_n})^\top$ and note that

$$\sum_{j,k} A_{jk} \cos(\theta_k - \theta_j) = \overline{q_\theta}^\top A q_\theta,$$

where A is the adjacency matrix, $\overline{q_\theta}$ is the complex conjugate of q_θ , and $^\top$ denotes the transpose of a vector. Since $\cos^2(\theta_k - \theta_j) = \frac{1}{2}(\cos(2(\theta_k - \theta_j)) + 1)$, we have

$$\sum_{j,k} A_{jk} \cos^2(\theta_k - \theta_j) = \frac{1}{2} \overline{q_{2\theta}}^\top A q_{2\theta} + \frac{1}{2} \mathbf{1}^\top A \mathbf{1},$$

where $\mathbf{1}$ is the vector of all ones and $q_{2\theta} = (e^{i2\theta_1}, \dots, e^{i2\theta_n})^\top$.

We would like to know when a stable equilibrium is close to the all-in-phase state, and we know this when ρ_1 is close to 1. Therefore, we begin by deriving a lower bound on ρ_1 for any stable equilibrium of (1). Similar inequalities involving ρ_1^2 and $|\rho_2|^2$ have been used to demonstrate that sufficiently dense Kuramoto networks are globally synchronizing.^{18,21}

Lemma 1. *Let G be a connected graph and θ a stable equilibrium of (1). For any $p > 0$, we have*

$$\rho_1^2 \geq \frac{1 + |\rho_2|^2}{2} - \frac{2\|\Delta_A\|}{np}, \quad (4)$$

where the mean-shifted adjacency matrix, Δ_A , is defined in (2).

Proof. Since $\Delta_A = A - pJ_n$, we have

$$\overline{q_\theta}^\top A q_\theta = p \overline{q_\theta}^\top J_n q_\theta + \overline{q_\theta}^\top \Delta_A q_\theta.$$

One finds that $\bar{q}_\theta^\top J_n q_\theta = n^2 \rho_1^2$ by (3) and that $|\bar{q}_\theta^\top \Delta_A q_\theta| \leq \|\Delta_A\| \|q_\theta\|^2 \leq n \|\Delta_A\|$; therefore,

$$\left| \sum_{j,k} A_{jk} \cos(\theta_k - \theta_j) - n^2 p \rho_1^2 \right| \leq n \|\Delta_A\|. \tag{5}$$

By the same reasoning for $\bar{q}_{2\theta}^\top A q_{2\theta}$, we find that

$$\left| \sum_{j,k} A_{jk} \cos(2(\theta_k - \theta_j)) - n^2 p |\rho_2|^2 \right| \leq n \|\Delta_A\|.$$

Moreover, $\mathbf{1}^\top A \mathbf{1} = \mathbf{1}^\top \Delta_A \mathbf{1} + n^2 p$; therefore, $|\mathbf{1}^\top A \mathbf{1} - n^2 p| \leq n \|\Delta_A\|$. We conclude that

$$\left| \sum_{j,k} A_{jk} \cos^2(\theta_k - \theta_j) - n^2 p \frac{|\rho_2|^2 + 1}{2} \right| \leq n \|\Delta_A\|. \tag{6}$$

Since θ is a stable equilibrium for (1), it is known that

$$\sum_{j,k} A_{jk} \cos(\theta_k - \theta_j) (1 - \cos(\theta_k - \theta_j)) \geq 0,$$

as shown by Ling *et al.*¹⁸ on p. 1893 of their paper. From (5) and (6), we must have

$$n^2 p \rho_1^2 + n \|\Delta_A\| \geq n^2 p \frac{|\rho_2|^2 + 1}{2} - n \|\Delta_A\|,$$

which is equivalent to the statement of the lemma. \square

To maximize the lower bound on ρ_1^2 from Lemma 1, one can optimize over p . For a random graph where each edge has a fixed probability of being present, independently of the other edges, we usually just select p to be that probability. Regardless, to make the lower bound on ρ_1^2 in Lemma 1 useful, we need to find a nontrivial lower bound on $|\rho_2|^2$ since this quantity appears in the right hand side of (4). To obtain such a bound, we use the following technical lemma.

Lemma 2. *Let G be a connected graph and θ a stable equilibrium. We have*

$$\|\Delta_A q_\theta\|^2 \geq n^2 p^2 \rho_1^2 \left(\sum_j \sin^2(\theta_j) + \sum_{j, \cos(\theta_j) \leq 0} \cos^2(\theta_j) \right),$$

where $\|\cdot\|$ is the Euclidean norm.

Proof. Select any j such that $1 \leq j \leq n$. From the fact that θ is an equilibrium, we have $\sum_k A_{jk} \sin(\theta_k - \theta_j) = 0$. Moreover, because the equilibrium is stable, we also have $\sum_k A_{jk} \cos(\theta_k - \theta_j) \geq 0$, which follows as the diagonal entries of the Hessian matrix must be nonnegative at a stable equilibrium [see (2.3) of Ling *et al.*¹⁸]. These inequalities can be written as

$$\operatorname{Re} \left(e^{-i\theta_j} e_j^\top A q_\theta \right) \geq 0 \quad \text{and} \quad \operatorname{Im} \left(e^{-i\theta_j} e_j^\top A q_\theta \right) = 0,$$

where e_j is the j th unit vector. Since $\Delta_A = A - pJ_n$ and using that $J_n q_\theta = n \rho_1 \mathbf{1}$, we find that

$$\left| \operatorname{Im} \left(e^{-i\theta_j} e_j^\top \Delta_A q_\theta \right) \right| = n p \rho_1 |\sin(\theta_j)|.$$

If $\cos \theta_j \leq 0$, then we also have

$$\begin{aligned} \operatorname{Re} \left(e^{-i\theta_j} e_j^\top \Delta_A q_\theta \right) &= \sum_k (A_{jk} - p) \cos(\theta_k - \theta_j) \\ &= \sum_k A_{jk} \cos(\theta_k - \theta_j) - n p \rho_1 \cos \theta_j \\ &\geq n p \rho_1 |\cos \theta_j|. \end{aligned}$$

The inequality in the lemma follows by squaring the above inequalities, summing over j , and noting that $|e^{-i\theta_j}| = 1$. \square

To see how Lemma 2 can be used to derive a lower bound on $|\rho_2|^2$ for any stable equilibrium state, we start by dropping the second sum in Lemma 2. Using the upper bound $\|\Delta_A q_\theta\|^2 \leq n \|\Delta_A\|^2$, we find that $\sum_j \sin^2(\theta_j) \leq \|\Delta_A\|^2 / (n p^2 \rho_1^2)$. Since $n \operatorname{Re}(\rho_2) = \sum_j \cos(2\theta_j) = \sum_j (1 - 2 \sin^2(\theta_j))$, we have the following lower bound on $|\rho_2|$:

$$|\rho_2| \geq \operatorname{Re}(\rho_2) = \frac{1}{n} \sum_j (1 - 2 \sin^2(\theta_j)) \geq 1 - \frac{2 \|\Delta_A\|^2}{n^2 p^2 \rho_1^2}. \tag{7}$$

From (4) and (7), we can deduce that when $\|\Delta_A\| \ll n p$, then ρ_1 and $|\rho_2|$ must both be close to 1. Intuitively, this should mean that the corresponding stable equilibrium, θ , must be close to the all-in-phase state with the possible exception of a small number of stray oscillators. However, our goal is to prove global synchrony, which is a more stringent condition, and we must completely rule out the existence of any stray oscillators.

To precisely control the number of stray oscillators, we define a set of indices for oscillators whose phases lie outside of a sector of half-angle ϕ (centered about the all-in-phase state),

$$C_\phi = \{k : \cos(\theta_k) \leq \cos(\phi), 1 \leq k \leq n\}, \tag{8}$$

for any angle $0 \leq \phi \leq \pi$ (see Fig. 1). If we can prove that $C_{\pi/2}$ is empty, then we know that all the phases in the equilibrium state lie strictly inside a half-circle. That would give us what we want because by a basic theorem for the homogeneous Kuramoto model, the only stable equilibrium θ with all its phases confined to a half-circle is the all-in-phase state.^{8,23} In terms of bounds, since $|C_{\pi/2}|$ is integer-valued, if we can show that $|C_{\pi/2}| < 1$, then we know that $C_{\pi/2}$ is the empty set.

From Lemma 2 and $\|\Delta_A q_\theta\|^2 \leq n \|\Delta_A\|^2$, we find that

$$\begin{aligned} n \|\Delta_A\|^2 &\geq n^2 p^2 \rho_1^2 \sum_{j \in C_\phi} \left(\sin^2(\theta_j) + \delta_{|\theta_j| > \pi/2} \cos^2(\theta_j) \right) \\ &\geq n^2 p^2 \rho_1^2 |C_\phi| \sin^2(\phi), \end{aligned} \tag{9}$$

where $\delta_{|\theta_j| > \pi/2}$ is 1 if $|\theta_j| > \pi/2$ and 0 otherwise. Therefore, by plugging in $\phi = \pi/2$, we see that $|C_{\pi/2}| \leq \|\Delta_A\|^2 / (n p^2 \rho_1^2)$. Thus, if $\|\Delta_A\|^2 < n p^2 \rho_1^2$, then $C_{\pi/2}$ must be the empty set and the network is globally synchronizing.

Unfortunately, this kind of reasoning is not sufficient to prove global synchrony for graphs of interest to us here. For example, for an Erdős-Rényi random graph, we know that $\|\Delta_A\|^2 \approx 4p(1-p)n$ with high probability for large n (see Sec. VI). Therefore, the upper bound on $|C_{\pi/2}|$ is approximately $4(1-p)/(p\rho_1^2)$, which for $p < 1/2$ is certainly not good enough to conclude that $C_{\pi/2}$ is empty. Instead,

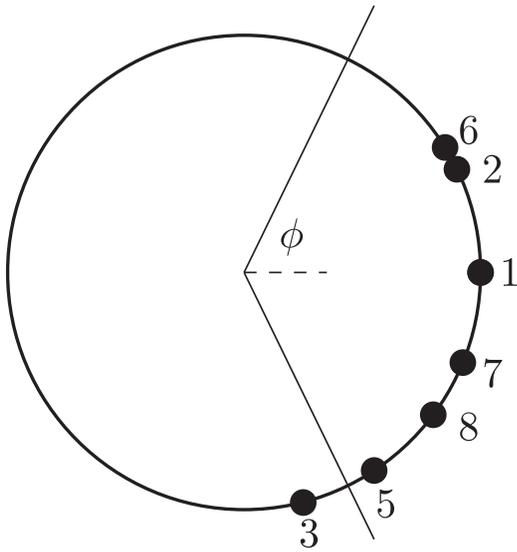


FIG. 1. When viewing the phases of the oscillators on the unit circle, the set C_ϕ contains the indices of stray oscillators whose phases have cosines less than or equal to $\cos(\phi)$. In the example shown here, we have eight oscillators, and only the oscillator with phase θ_3 has strayed outside the sector of half-angle ϕ . Hence, $C_\phi = \{3\}$.

we must further improve our bounds on $|C_{\pi/2}|$ by using a recursive refinement strategy that we refer to as an “amplification” argument (see Sec. V).

IV. BOUNDS ON THE NUMBER OF EDGES AND SIZES OF SETS IN GRAPHS

The precise amplification argument that we use requires bounds on the number of edges and sizes of vertex sets of a graph expressed in terms of the spectral norms of Δ_A and Δ_L . It is worth noting that these bounds hold for both deterministic and random graphs. For a vertex set C of a graph G , we denote the characteristic vector by v_C ; i.e., $(v_C)_j = 1$ if $j \in C$ and $(v_C)_j = 0$ if $j \notin C$. We use v_C^T to denote the vector transpose of v_C . We write $|C|$ to be the number of vertices in C and denote the number of (directed) edges in G starting at vertex set C and ending at C' as $E_{C,C'}$. Therefore, $E_{C,C}$ is twice the number of (undirected) edges between vertices in C . For Erdős–Rényi graphs, one expects to have $E_{C,C'} \approx p|C||C'|$. However, for our argument to work, expectations are not adequate; instead, we need to have bounds on the difference between $E_{C,C'}$ and $p|C||C'|$. The results in this section are proved by classical techniques, and closely related bounds are well known. We give the proofs of the precise inequalities that we need so that the paper is self-contained.

We first show that for any vertex set C , the number of edges connecting a vertex in C to another vertex in C deviates from $p|C|^2$ by at most $\|\Delta_A\||C|$ for any $0 \leq p \leq 1$.

Lemma 3. *Let G be a graph of size n with vertex set V_G and adjacency matrix A . For any $0 \leq p \leq 1$, we have*

$$\max_{C \subseteq V_G} \frac{|E_{C,C} - p|C|^2|}{|C|} \leq \|\Delta_A\|,$$

where Δ_A is defined in (2).

Proof. Let v_C be the characteristic vector for C . By the min-max theorem,³⁵ we have $\max_{C \subseteq V_G} (v_C^T \Delta_A v_C) / (v_C^T v_C) \leq \|\Delta_A\|$. Finally, we note that $\Delta_A = A - pJ_n$, $v_C^T v_C = |C|$, $v_C^T J_n v_C = |C|^2$, and $v_C^T A v_C = E_{C,C}$. \square

Lemma 3 controls the number of edges connecting a set of vertices. The following result controls the number of edges leaving a vertex set. We denote the vertices of G that are not in C_1 by $V_G \setminus C_1$.

Lemma 4. *Let G be a graph of size n with vertex set V_G and graph Laplacian L . For any $0 \leq p \leq 1$, we have*

$$\max_{C_1 \subseteq V_G} \frac{|E_{C_1, \overline{C_1}} - p|C_1||\overline{C_1}||}{|C_1||\overline{C_1}|} \leq \frac{\|\Delta_L\|}{n},$$

where Δ_L is defined in (2) and $\overline{C_1} = V_G \setminus C_1$.

Proof. Let v_{C_1} and $v_{\overline{C_1}}$ be the characteristic vectors for the sets C_1 and $\overline{C_1}$, respectively, and $w = (|\overline{C_1}|/n)v_{C_1} - (|C_1|/n)v_{\overline{C_1}}$. By the min-max theorem,³⁵ we have

$$\max_{C_1 \subseteq V_G} \frac{w^T \Delta_L w}{w^T w} \leq \|\Delta_L\|.$$

Finally, we note that $\Delta_L = L + pJ_n - npI_n$, $w^T w = \frac{|\overline{C_1}|^2|C_1|}{n^2} + \frac{|C_1||C_1|^2}{n^2} = \frac{|\overline{C_1}||C_1|}{n}$ as $|C_1| + |\overline{C_1}| = n$, $w^T (pJ_n - npI_n)w = -p|C_1||\overline{C_1}|$, and $w^T Lw = E_{C_1, \overline{C_1}}$. \square

When a bound on $\|\Delta_L\|$ is available, Lemma 4 can be used to ensure that G is connected. In particular, Lemma 4 tells us that a graph of size n is connected if $\|\Delta_L\| < np$ for some $0 \leq p \leq 1$.

Since $0 \leq E_{C_1, \overline{C_1}} \leq |C_1||\overline{C_1}|$, we know that $|E_{C_1, \overline{C_1}} - p|C_1||\overline{C_1}|| \leq \max\{p, 1 - p\}|C_1||\overline{C_1}|$. Therefore, Lemma 4 is only a useful bound when $\|\Delta_L\| \ll np$. We can take Lemma 4 a step further and bound the number of edges between any two disjoint sets of vertices of G using $\|\Delta_L\|$. We denote the union of the vertex sets C_1 and C_2 by $C_1 \sqcup C_2$.

Lemma 5. *Let G be a graph of size n with vertex set V_G and graph Laplacian matrix L . For any $0 \leq p \leq 1$, we have*

$$\max_{\substack{C_1 \subseteq V_G, C_2 \subseteq V_G \setminus C_1, \\ C_3 = V_G \setminus (C_1 \cup C_2)}} \frac{|E_{C_1, C_2} - p|C_1||C_2||}{|C_1||C_2| + |C_1||C_3| + |C_2||C_3|} \leq \frac{\|\Delta_L\|}{n},$$

where Δ_L is defined in (2).

Proof. For any partitioning of the vertices of G into three sets C_1 , C_2 , and C_3 , we have $2E_{C_1, C_2} = E_{C_1, C_2 \cup C_3} + E_{C_2, C_1 \cup C_3} - E_{C_3, C_1 \cup C_2}$. By Lemma 4, $E_{C_1, C_2 \cup C_3}$ is bounded between $p(|C_1||C_2| + |C_1||C_3|) \pm \|\Delta_L\||C_1|(|C_2| + |C_3|)/n$, $E_{C_2, C_1 \cup C_3}$ is bounded between $p(|C_2||C_1| + |C_2||C_3|) \pm \|\Delta_L\||C_2|(|C_1| + |C_3|)/n$, and $E_{C_3, C_1 \cup C_2}$ is bounded between $p(|C_3||C_1| + |C_3||C_2|) \pm \|\Delta_L\||C_3|(|C_1| + |C_2|)/n$. Hence, E_{C_1, C_2} deviates from $p|C_1||C_2|$ by less than $\|\Delta_L\|(|C_1||C_2| + |C_2||C_3| + |C_1||C_3|)/(2n)$. \square

Lemmas 3 and 5 can be combined to obtain the following result. We do not regard Theorem 6 as having significant independent interest, but later, it is an important statement in the proof of Corollary 9. It only deserves the status of a theorem in the sense that it is a key technical step in our overall argument.

Theorem 6. Let G be a graph of size n with adjacency matrix A and graph Laplacian matrix L . Suppose that C_1, C_2 , and C_3 are a partition of the vertices of G into three sets such that (i) $E_{C_1, C_3} \leq \lambda E_{C_3, C_3}$ for some number λ and (ii) $|C_2| < |C_1|$. Then, for any $0 \leq p \leq 1$, we have

$$|C_2| + |C_3| \geq \left(\frac{n(p|C_1| - p\lambda|C_3| - \lambda\|\Delta_A\|)}{\|\Delta_L\||C_1|} - 1 \right) |C_3|,$$

where Δ_A and Δ_L are defined in (2).

Proof. By Lemma 3, E_{C_3, C_3} deviates from $p|C_3|^2$ by less than $\|\Delta_A\||C_3|$, and, by Lemma 5, E_{C_1, C_3} deviates from $p|C_1||C_3|$ by less than $\|\Delta_L\|(|C_1||C_2| + |C_2||C_3| + |C_3||C_1|)/n$. Since $E_{C_1, C_3} \leq \lambda E_{C_3, C_3}$, we must have

$$p|C_1||C_3| - \frac{\|\Delta_L\|}{n}(|C_1||C_2| + |C_2||C_3| + |C_3||C_1|) \leq \lambda(p|C_3|^2 + \|\Delta_A\||C_3|).$$

By rearranging this inequality, we find that

$$|C_2| + |C_3| \geq \left(\frac{n(p|C_1| - p\lambda|C_3| - \lambda\|\Delta_A\|)}{\|\Delta_L\||C_1|} - \frac{|C_2|}{|C_1|} \right) |C_3|.$$

The result follows as $|C_2| < |C_1|$. □

V. AMPLIFICATION ARGUMENT

We are finally ready for our amplification argument, which is a way to improve the bounds on $|C_{\pi/2}|$ from (9). We first write down a new inequality that holds for any stable equilibrium. We write it down using a kernel function K that later allows us to improve our argument with the aid of a computer (see Sec. IV D). For any stable equilibrium $\theta = (\theta_1, \dots, \theta_n)$ of (1), we know that it satisfies equilibrium and stability conditions. The equilibrium conditions are given by $\sum_j A_{jk} \sin(\theta_k - \theta_j) = 0$ for all $1 \leq k \leq n$. In addition, stability requires that the Hessian matrix [see (2.3) of Ling *et al.*¹⁸] associated with θ is a nonnegative definite matrix; i.e., it has nonnegative eigenvalues.

The fact that the Hessian matrix has nonnegative eigenvalues can be used to form many useful inequalities. For example, if $H_A(\theta)$ is the Hessian matrix, then a stable equilibrium must satisfy the Rayleigh quotient $x^T H_A(\theta)x \geq 0$ for any $n \times 1$ vector x . In particular, selecting $x = e_k$ (the k th unit vector) implies that the k th diagonal entry of $H_A(\theta)$ is nonnegative. This yields the inequalities $e_k^T H_A(\theta)e_k = \sum_j A_{jk} \cos(\theta_k - \theta_j) \geq 0$ for $1 \leq k \leq n$. Next, one can combine $\sum_j A_{jk} \sin(\theta_k - \theta_j) = 0$ and $\sum_j A_{jk} \cos(\theta_k - \theta_j) \geq 0$ in tricky ways and then use trigonometric identities to simplify the inequalities. After manipulating these inequalities for a while, we find that it is beneficial to massage them differently depending on the values of θ_k and θ_j that appear in them. However, then, having combined the inequalities in diverse ways, we notice that they all share a common structure. To highlight this commonality, we express the various inequalities as a unified family of the form $\sum_j A_{jk} K(\theta_j, \theta_k) \geq 0$, where K is defined piecewise to reflect its dependence on its arguments.

Lemma 7. Let G be a graph with adjacency matrix A . Let K be a kernel function defined on $[-\pi, \pi) \times [-\pi, \pi)$, given by

$$K(\alpha, \beta) = \begin{cases} \sin(|\alpha| - |\beta|), & |\alpha|, |\beta| \leq \frac{\pi}{2}, \\ -\cos(\alpha), & |\alpha| \leq \frac{\pi}{2}, |\beta| > \frac{\pi}{2}, \\ 1, & |\alpha| > \frac{\pi}{2}. \end{cases}$$

Then, any stable equilibrium θ of (1) must satisfy $\sum_j A_{jk} K(\theta_j, \theta_k) \geq 0$ for any $1 \leq k \leq n$.

Proof. Let k be an integer between 1 and n . Due to periodicity, we may assume that the phases, θ_j , take values in the interval $[-\pi, \pi)$. We split the proof into three cases depending on the value of θ_k .

Case 1: $0 \leq \theta_k \leq \pi/2$. We first show that $\sin(\theta_j - \theta_k) \leq K(\theta_j, \theta_k)$ for all j by checking the three possible subcases: (i) If $0 \leq \theta_j \leq \pi/2$, then $\sin(\theta_j - \theta_k) = \sin(|\theta_j| - |\theta_k|) = K(\theta_j, \theta_k)$. (ii) If $|\theta_j| > \pi/2$, then $\sin(\theta_j - \theta_k) \leq 1 = K(\theta_j, \theta_k)$. (iii) If $-\pi/2 \leq \theta_j < 0$, then $\sin(\theta_j - \theta_k) = \sin(|\theta_j| - |\theta_k| - 2|\theta_j|) \leq \sin(|\theta_j| - |\theta_k|) = K(\theta_j, \theta_k)$, where the inequality holds because $-\pi/2 \leq |\theta_j| - |\theta_k| \leq \pi/2$ and $0 \leq 2|\theta_j| \leq \pi$. The inequality follows from the equilibrium condition $\sum_j A_{jk} \sin(\theta_j - \theta_k) = 0$.

Case 2: $-\pi/2 \leq \theta_k < 0$. For this case, we begin by showing that $\sin(\theta_j - \theta_k) \geq -K(\theta_j, \theta_k)$ for all j by checking the three possible subcases: (i) If $0 \leq \theta_j \leq \pi/2$, then $\sin(\theta_j - \theta_k) = \sin(|\theta_j| - |\theta_k| + 2|\theta_k|) \geq -\sin(|\theta_j| - |\theta_k|) = -K(\theta_j, \theta_k)$, where the inequality holds because $-\pi/2 \leq |\theta_j| - |\theta_k| \leq \pi/2$ and $0 \leq 2|\theta_k| \leq \pi$. (ii) If $|\theta_j| > \pi/2$, then $\sin(\theta_j - \theta_k) \geq -1 = -K(\theta_j, \theta_k)$. (iii) If $-\pi/2 \leq \theta_j < 0$, then $\sin(\theta_j - \theta_k) = \sin(-|\theta_j| + |\theta_k|) = -\sin(|\theta_j| - |\theta_k|) = -K(\theta_j, \theta_k)$. The inequality follows from the equilibrium condition $\sum_j A_{jk} \sin(\theta_j - \theta_k) = 0$.

Case 3: $|\theta_k| > \pi/2$. From the fact that θ is an equilibrium, we have $\sum_j A_{jk} \sin(\theta_k - \theta_j) = 0$. Moreover, $\sum_j A_{jk} \cos(\theta_k - \theta_j) \geq 0$ because the diagonal entries of the Hessian matrix must be nonnegative at a stable equilibrium [see (2.3) of Ling *et al.*¹⁸]. By a trigonometric identity, we find that $0 = \sum_k A_{jk} \sin(\theta_k - \theta_j) = \sin(\theta_k) \sum_j A_{jk} \cos(\theta_j) + \cos(\theta_k) \sum_j A_{jk} \sin(\theta_j)$, and, hence,

$$\sum_j A_{jk} \sin(\theta_j) = \frac{\sin(\theta_k)}{\cos(\theta_k)} \sum_j A_{jk} \cos(\theta_j), \tag{10}$$

where we note that $\cos(\theta_k) \neq 0$ as $\theta_k \neq \pm\pi/2$. Moreover, from another trigonometric identity, we have $0 \leq \sum_j A_{jk} \cos(\theta_k - \theta_j) = \cos(\theta_k) \sum_j A_{jk} \cos(\theta_j) + \sin(\theta_k) \sum_j A_{jk} \sin(\theta_j)$, which together with (10) gives

$$\left(\cos(\theta_k) + \frac{\sin^2(\theta_k)}{\cos(\theta_k)} \right) \sum_j A_{jk} \cos(\theta_j) \geq 0.$$

Multiplying this inequality by $\cos(\theta_k)$ [note that $\cos(\theta_k) < 0$ as $|\theta_k| > \pi/2$] and using $\cos^2(\theta_k) + \sin^2(\theta_k) = 1$, we conclude that $\sum_j A_{jk} \cos(\theta_j) \leq 0$. To reach the desired inequality in the statement of the lemma, we now check the two possible subcases: (i) If $|\theta_j| \leq \pi/2$, then $\cos(\theta_j) = -K(\theta_j, \theta_k)$. (ii) If $|\theta_j| > \pi/2$, then $\cos(\theta_j) \geq -1 = -K(\theta_j, \theta_k)$. This means that $0 \geq \sum_j A_{jk} \cos(\theta_j) \geq -\sum_j A_{jk} K(\theta_j, \theta_k)$ as desired. □

In preliminary work, we have found indications that Lemma 7 can be used to prove stronger results than those we report below;

see Sec. IV D for further discussion. However, we are not sure yet how to write down an argument that uses the full strength of Lemma 7 in a readily digestible fashion. So for now, we use the following simplified lemma instead. It is a key step in our amplification argument.

Lemma 7 shows us that if θ is a stable equilibrium, then any oscillator with phase $> \pi/2$ in absolute value cannot be coupled to too many oscillators with phases $< \pi/2$. This is because $K(\alpha, \beta) = -\cos(\alpha) < 0$ when $|\alpha| < \pi/2$ and $|\beta| > \pi/2$. If there are too many negative contributions in the sum $\sum_j A_{jk}K(\theta_j, \theta_k)$, then it will end up negative, which is not allowed for a stable equilibrium. To make this precise, we can use our sets C_β in (8).

Lemma 8. *Let G be a graph and θ a stable equilibrium of (1), and let $0 < \alpha < \beta < \pi/2$. We have*

$$E_{C_\beta, C_\beta} \geq \sin(\beta - \alpha) E_{C_\beta, \overline{C_\alpha}},$$

where C_α and C_β are defined in (8). Here, $\overline{C_\alpha} = V_G \setminus C_\alpha$.

Proof. This follows from the previous lemma by carefully bounding $K(\theta_j, \theta_k)$ when $k \in C_\beta$ (which implies that $|\theta_k| > \beta$). We check the three possible subcases: (i) If $j \in C_\beta$, then we might have $|\theta_j| > \pi/2$; therefore, the best we can say is $K(\theta_j, \theta_k) \leq 1$. (ii) If $j \in C_\alpha \setminus C_\beta$ (which implies that $|\theta_j| \leq \pi/2$), then either $|\theta_k| > \pi/2$ so that $K(\theta_j, \theta_k) = -\cos(\theta_j) \leq 0$ or $\beta < |\theta_k| \leq \pi/2$ so that $K(\theta_j, \theta_k) = \sin(|\theta_j| - |\theta_k|) \leq 0$ as $|\theta_j| \leq |\theta_k|$. Either way, we have $K(\theta_j, \theta_k) \leq 0$ when $j \in C_\alpha \setminus C_\beta$. (iii) If $j \notin C_\alpha$ (which implies that $|\theta_j| < \alpha$), then either $|\theta_k| > \pi/2$ so that $K(\theta_j, \theta_k) = -\cos(\theta_j) \leq -\cos(\alpha) = -\sin(\pi/2 - \alpha) \leq -\sin(\beta - \alpha)$ or $\beta < |\theta_k| \leq \pi/2$ so that $K(\theta_j, \theta_k) = \sin(|\theta_j| - |\theta_k|) \leq -\sin(\beta - \alpha)$. Either way, $K(\theta_j, \theta_k) \leq -\sin(\beta - \alpha)$ when $j \notin C_\alpha$. We conclude that

$$K(\theta_j, \theta_k) \leq \begin{cases} 1, & j \in C_\beta, \\ 0, & j \in C_\alpha \setminus C_\beta, \\ -\sin(\beta - \alpha), & j \notin C_\alpha, \end{cases}$$

and, hence, by Lemma 7, we have

$$0 \leq \sum_{k \in C_\beta} \sum_j A_{jk} K(\theta_j, \theta_k) \leq \underbrace{\sum_{j, k \in C_\beta} A_{jk}}_{=E_{C_\beta, C_\beta}} - \sin(\beta - \alpha) \underbrace{\sum_{j \notin C_\alpha, k \in C_\beta} A_{jk}}_{=E_{C_\beta, \overline{C_\alpha}}}$$

as desired. \square

We now show that if θ is a stable equilibrium and $|C_\alpha|$ is small, then $|C_\beta|$ must be even smaller for $0 < \alpha < \beta < \pi/2$; otherwise, the oscillators in the set $C_\alpha \setminus C_\beta$ would destabilize the equilibrium (see Fig. 2). Since we know that $|C_\beta| \leq |C_\alpha|$, the next bound is useful when $\|\Delta_L\|/(np) > 1/4$.

Corollary 9. *Let G be a graph of size n , θ a stable equilibrium of (1), and $0 < p \leq 1$. If for some $0 < \alpha < \beta < \pi/2$, we have $|C_\alpha| \leq n/2$, $|C_\beta| \leq 2\|\Delta_A\|/p$, and $\sin(\beta - \alpha) \geq 12\|\Delta_A\|/(np)$, then*

$$|C_\beta| \leq \left(\frac{np}{2\|\Delta_L\|} - 1 \right)^{-1} |C_\alpha|,$$

where Δ_A and Δ_L are defined in (2) and C_α and C_β in (8).

Proof. Let $\lambda = np/(12\|\Delta_A\|)$. By Lemma 8, we have $E_{C_\beta, \overline{C_\alpha}} \leq (1/\sin(\beta - \alpha))E_{C_\beta, C_\beta} \leq \lambda E_{C_\beta, C_\beta}$. Hence, by Theorem 6 (with

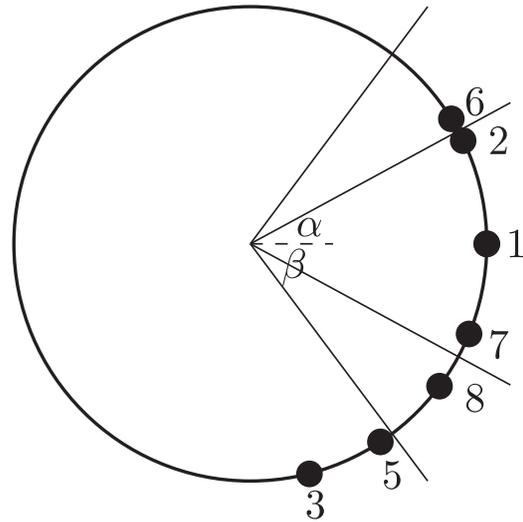


FIG. 2. Here, $C_\alpha = \{3, 5, 6, 8\}$ and $C_\beta = \{3, 5\}$. We illustrate a hypothetical equilibrium such that $C_{\pi/2}$ is empty and, therefore, must be unstable; however, it might be that (9) is not tight enough to show that $|C_{\pi/2}| < 1$. To still conclude that $C_{\pi/2}$ is empty, we first show in Lemma 8 that for an equilibrium to be stable, there must be enough edges coupling the oscillators in C_β together, compared to those between C_β and $\overline{C_\alpha}$. For the illustrated equilibrium, we are comparing the number of internal edges between oscillators 3 and 5 and the number of outgoing edges to oscillators 1, 2, and 7.

$C_1 = \overline{C_\alpha}$, $C_2 = C_\alpha \setminus C_\beta$, and $C_3 = C_\beta$), we find that

$$\begin{aligned} |C_\alpha| &\geq \left(\frac{n(p|\overline{C_\alpha}| - p\lambda|C_\beta| - \lambda\|\Delta_A\|)}{\|\Delta_L\||\overline{C_\alpha}|} - 1 \right) |C_\beta| \\ &= \left(\frac{np}{\|\Delta_L\|} - \frac{n\lambda(p|C_\beta| + \|\Delta_A\|)}{\|\Delta_L\||\overline{C_\alpha}|} - 1 \right) |C_\beta| \\ &\geq \left(\frac{np}{2\|\Delta_L\|} - 1 \right) |C_\beta|, \end{aligned}$$

where the last inequality holds if $\lambda \leq \frac{\|\Delta_L\||\overline{C_\alpha}|}{n(p|C_\beta| + \|\Delta_A\|)} \left(\frac{np}{2\|\Delta_L\|} \right)$. Since $|\overline{C_\alpha}| \geq n/2$ and $|C_\beta| \leq 2\|\Delta_A\|/p$, we find that $\lambda = np/(12\|\Delta_A\|)$ satisfies this upper bound. \square

Note that it is only possible to have an α and β such that $\sin(\beta - \alpha) \geq 12\|\Delta_A\|/(np)$ in Corollary 9 when $\|\Delta_A\|/(np) < 1/12$. Corollary 9 can be used in a recursive fashion to improve the bound on $|C_{\pi/2}|$. Below, we start at α and incrementally increase β to conclude that $C_{\pi/2}$ is empty.

Lemma 10. *Let G be a graph of size n , θ a stable equilibrium of (1), $0 < p \leq 1$, $\|\Delta_A\|/(np) < 1/12$, and $\|\Delta_L\|/(np) < 1/4$. If for some $\alpha < \pi/2$, we have (i) $|C_\alpha| < 2\|\Delta_A\|/p$ and (ii),*

$$\frac{\pi/2 - \alpha}{\sin^{-1}\left(\frac{12\|\Delta_A\|}{np}\right)} > \frac{\log(|C_\alpha|)}{\log\left(\frac{np}{2\|\Delta_L\|} - 1\right)} + 1,$$

then $C_{\pi/2}$ is empty and θ is the all-in-phase state. [Here, if C_α is empty, then take $|C_\alpha| = 0$ and $\log(0) = -\infty$.]

Proof. Set $\beta_k = \alpha + k \sin^{-1}(12\|\Delta_A\|/(np))$. Since we need $\beta_k < \pi/2$, we can take $0 \leq k \leq M$, where

$$M = \left\lceil \frac{\pi/2 - \alpha}{\sin^{-1}\left(\frac{12\|\Delta_A\|}{np}\right) - 1} \right\rceil.$$

By Corollary 9 and the fact that $|C_\alpha| < 2\|\Delta_A\|/p < n/6$ (which ensures that $|C_\alpha| \leq n/2$), we have

$$\begin{aligned} |C_{\beta_M}| &\leq \left(\frac{np}{2\|\Delta_L\|} - 1\right)^{-1} |C_{\beta_{M-1}}| \\ &\vdots \\ &\leq \left(\frac{np}{2\|\Delta_L\|} - 1\right)^{-M} |C_\alpha|. \end{aligned}$$

Since $|C_{\pi/2}| \leq |C_{\beta_M}|$ to conclude that $|C_{\pi/2}| < 1$, we need $(np/(2\|\Delta_L\|) - 1)^{-M} |C_\alpha| < 1$. This is guaranteed when $M > \log(|C_\alpha|/\log(np/(2\|\Delta_L\|) - 1))$ and the result follows. \square

Finally, we summarize our findings. In particular, we can now provide a list of technical criteria that ensure that the network is globally synchronizing.

Theorem 11. *Let G be a graph with n vertices and $0 < p < 1$.*

If

- (i) $\|\Delta_A\|/(np) < 1/12$,
- (ii) $\|\Delta_L\|/(np) < 1/4$, and
- (iii)

$$\frac{\pi/4}{\sin^{-1}\left(\frac{12\|\Delta_A\|}{np}\right)} > \frac{\log(n/6)}{\log\left(\frac{np}{2\|\Delta_L\|} - 1\right)} + 1,$$

then G is globally synchronizing.

Proof. Let θ be any stable equilibrium of (1) on G . By combining (4) and (7), we find that

$$\rho_1^6 - (1 - 2a)\rho_1^4 + 2a^2\rho_1^2 - 2a^4 \geq 0, \tag{11}$$

where $a = \|\Delta_A\|/np$. Since $a < 1/12$ by (i), one can check that (11) implies that $\rho_1^2 > a$. [In fact, if $a < 1/5$, then (11) implies that $\rho_1^2 > a$.] Now, select $\phi = \pi/4$. By (9), we find that $|C_{\pi/4}| \leq 2\|\Delta_A\|^2/(np^2\rho_1^2) < 2\|\Delta_A\|/p$ as $\rho_1^2 > \|\Delta_A\|/(np)$. Hence, we find that $|C_{\pi/4}| \leq n/6$. By taking $\alpha = \pi/4$ in Lemma 10 and as (ii) holds, we find that θ is the all-in-phase state when (iii) is satisfied. \square

Theorem 11 shows that a graph’s global synchrony can be ensured by the size of $\|\Delta_A\|$ and $\|\Delta_L\|$ alone. This is particularly beneficial for random networks as $\|\Delta_A\|$ and $\|\Delta_L\|$ are quantities that are studied in the random matrix literature. For random networks, hypotheses (i)–(iii) become inequalities involving random variables, namely, $\|\Delta_A\|$ and $\|\Delta_L\|$. It is crucial to appreciate that these inequalities are not *independent* of each other, and we must be careful about this dependence when applying Theorem 11.

VI. THE GLOBAL SYNCHRONY OF ERDŐS-RÉNYI GRAPHS

The random matrix literature involves a large array of bounds on $\|\Delta_A\|$ and $\|\Delta_L\|$, and navigating the literature can be daunting

to outsiders. The main thing to keep in mind is that some of these bounds are asymptotic in nature and only hold for enormously large n , while others are explicit (but weaker) in n . We find that both kinds of bounds are useful. If we want to say something about a random graph for a given n , then we need explicit bounds; whereas if we are interested in the asymptotic properties, then we may use either explicit or asymptotic bounds.

A. Explicit bounds to derive explicit statements

We begin with a pair of explicit bounds on $\|\Delta_A\|$ and $\|\Delta_L\|$ for Erdős–Rényi random graphs with probability $0 < p < 1$. In 2019, Ling *et al.*¹⁸ established that (also, see Theorem 6.6.1 of Ref. 36)

$$\mathbb{P}[\|\Delta_A\| \geq f(n, p)] < 2n^{-1}, \quad \mathbb{P}[\|\Delta_L\| \geq 2f(n, p)] < 2n^{-1}, \tag{12}$$

where $f(n, p) = 2\sqrt{n \log(n)p(1-p)} + 4 \log(n)/3$. By applying the union bound to the pair of inequalities in (12), we conclude that *both* $\|\Delta_A\| \geq f(n, p)$ and $\|\Delta_L\| \geq f(n, p)$ hold with probability $> 1 - 4/n$.

Next, we use this conclusion to prove that an Erdős–Rényi graph for a particular n and p globally synchronizes with high probability. For example, let $n = 10^6$. Then, we try a range of p from $0 < p < 1$ and test for which p all three hypotheses in Theorem 11 hold. If the three hypotheses all hold, then the graph is globally synchronizing. By following this procedure, we find that for $p > 0.256$, the corresponding Erdős–Rényi graph is globally synchronizing with probability $> 1 - 4/n$. More specifically, when $n = 10^6$ and $p > 0.256$, one can calculate that $f(n, p)/(np) < 1/12$ and $2f(n, p)/(np) < 1/4$. Hence, with probability $> 1 - 4/n = 0.999996$, hypotheses (i) and (ii) in Theorem 11 hold. [In fact, hypotheses (i) and (ii) hold for much lower values of p , and hypothesis (iii) is the reason that we need $p > 0.256$.]

It is at this point that we need to be careful about the lack of independence of the three hypotheses in Theorem 11. In particular, we do not check (iii) directly as this would involve messy conditional probability calculations. Instead, we find it easier to check hypothesis (iii) by checking the following deterministic inequality:

$$\frac{\pi/4}{\sin^{-1}\left(\frac{12f(n,p)}{np}\right)} > \frac{\log(n/6)}{\log\left(\frac{np}{4f(n,p)} - 1\right)} + 1. \tag{13}$$

This deterministic inequality implies (iii) provided that $\|\Delta_A\| \leq f(n, p)$ and $\|\Delta_L\| \leq f(n, p)$, which we already know hold simultaneously with probability $> 1 - 4/n$. Next, using Theorem 11 with $f(n, p)$ replaced by $\|\Delta_A\|$ and $\|\Delta_L\|$ on the left and right of the inequality, respectively, we conclude that an Erdős–Rényi graph with $n = 10^6$ and $p > 0.256$ is globally synchronizing with probability > 0.999996 . By the same argument, we find that for $n = 10^7$, $p > 0.0474$ suffices.

B. Explicit bounds to derive asymptotic statements

Next, we leverage the explicit probability bounds in (12) to derive an asymptotic statement about the global synchrony of Erdős–Rényi graphs.

Theorem 12. *An Erdős–Rényi graph of size n is globally synchronizing with high probability $> 1 - 4/n$ provided that*

$$p \gg \log^3(n)/n.$$

Proof. If we can establish that all three hypotheses in Theorem 11 hold, then the Erdős–Rényi graph is globally synchronizing. As before, by applying the union bound to the pair of inequalities in (12), we conclude that both $\|\Delta_A\| \leq f(n, p)$ and $\|\Delta_L\| \leq f(n, p)$ hold with probability $> 1 - 4/n$. Then, by taking p of the form $p = c \log^\gamma(n)/n$ for some $c > 0$ and $\gamma > 1$, we see that $f(n, p) = O(\log^{1/2+\gamma/2}(n))$ as $n \rightarrow \infty$. Hence, we find that $f(n, p)/(np) = O(\log^{1/2-\gamma/2}(n))$, which shrinks to 0 as $n \rightarrow \infty$. We conclude that hypotheses (i) and (ii) hold for any $\gamma > 1$ provided (12) holds. For hypothesis (iii) in Theorem 11, we again check the deterministic inequality in (13). Since $\sin^{-1}(x) \approx x$ for small x , we see that the left hand side of (13) grows asymptotically like $\log^{-1/2+\gamma/2}(n)$, while the right hand side grows like $\log(n)/\log(\log^{-1/2+\gamma/2}(n))$. We find that $\gamma \geq 3$ is sufficient to ensure that $\log^{-1/2+\gamma/2}(n) \gg \log(n)/\log(\log^{-1/2+\gamma/2}(n))$, and the statement of the theorem then follows by applying Theorem 11. \square

C. Asymptotic bounds to derive asymptotic statements

Next, we turn to the stronger asymptotic probability bounds. There are many potentially useful asymptotic bounds in the work of Füredi and Komlós,²⁵ Vu,²⁶ Feige and Ofek,³⁷ and Lei and Rinaldo.³⁸ Taken together, this literature shows that with high probability, we have $\|\Delta_A\| = O(\sqrt{np})$ (see Theorem 5.2 in Ref. 38) and $\|\Delta_L\| = O(\sqrt{np(1-p)})$ (see Ref. 37). By the same argument as in the proof of Theorem 12, but with these stronger probability bounds, we arrive at our main result.

Theorem 13. *An Erdős–Rényi graph of size n is globally synchronizing with probability $> 1 - 4/n$ provided that*

$$p \gg \log^2(n)/n.$$

D. Using explicit bounds and optimizing with a computer

We now return to using the explicit bounds in (12) and aim to optimize our amplification argument using a computer. For a given n , we find that one can significantly improve the range of p for which the corresponding Erdős–Rényi graph is globally synchronizing (see Table I). Our computer program can be turned into a proof, and thus, for p above the thresholds in Table I, the Erdős–Rényi networks are globally synchronizing with probability $> 1 - 4/n$. However, writing the proofs down is unwieldy since the program works with bounds on $|C_\phi|$ for 1000 different values of ϕ and iteratively refines those bounds 100 000 times over.

By starting with $\rho_1 = 0$ and $\rho_2 = 0$, one can alternate between (4) and (7)—in an iterative fashion—to obtain a lower bound on ρ_1 . The lower bound on ρ_1 can be substituted in (9) to give initial upper bounds on $|C_\phi|$ for $0 \leq \phi \leq \pi/2$. One can then use Corollary 9 to progressively improve the bounds on $|C_\phi|$ for $0 \leq \phi \leq \pi/2$. (Technically, we are using a slight strengthening of Corollary 9, where we redo the proof keeping all the inequalities as tight as possible. However, using Corollary 9 gives extremely similar results.) To do so, one selects $0 < \alpha < \beta < \pi/2$ and attempts to apply Corollary 9. If the application of Corollary 9 is successful, then one also has $|C_\phi| \leq |C_\beta|$ for $\beta \leq \phi \leq \pi/2$. Since $|C_\phi|$ is integer-valued, any

TABLE I. The values of p in the Erdős–Rényi random graph model for which we can prove global synchrony for $n = 10^4, 10^5, 10^6$, and 10^7 with probability $> 1 - 4/n$. We used a computer to recursively apply inequalities in our paper to obtain refined bounds on $|C_{\pi/2}|$. We include this table to demonstrate that our results are meaningful for Erdős–Rényi graphs of finite, practical size. It is possible that these lower bounds on p can be improved by careful optimizations.

n	10^4	10^5	10^6	10^7
p	$>0.332\ 37$	$>0.071\ 68$	$>0.011\ 17$	$>0.001\ 57$

upper bound that is < 1 implies that C_ϕ is empty. We repeat this procedure 100 000 times to refine the upper bounds on $|C_\phi|$ for $0 \leq \phi \leq \pi/2$. If at any point we have $|C_{\pi/2}| < 1$, then we conclude that the Erdős–Rényi graph is globally synchronizing with probability $> 1 - 4/n$. In Table I, we used the explicit value of $f(n, p) = 2\sqrt{n} \log(n) p(1-p) + 4 \log(n)/3$ to bound the spectral norms of Δ_A and Δ_L .

There are several further improvements to our computer program that we tried: (1) Using stronger probability bounds^{25,26,37,38} for $\|\Delta_A\|$; (2) using Lemma 7 instead of Lemma 8, and (3) doing additional optimizations to improve the upper bounds for $|C_\phi|$. For example, by selecting triples $0 < \alpha < \beta_1 < \beta_2 < \pi/2$ and proving a generalization of Corollary 9, we get bounds for $|C_{\beta_1}|$ in terms of $|C_\alpha|$ and $|C_{\beta_2}|$ and bounds for $|C_{\beta_2}|$ in terms of $|C_\alpha|$ and $|C_{\beta_1}|$. There are similar generalizations for more than three angles. For example, when $n = 10^{20}$, using Lemma 8, we can only show that $p > 1.58 \times 10^{-15}$ guarantees that an Erdős–Rényi graph is globally synchronizing with high probability, but with these extra improvements, we find that $p > 3.50 \times 10^{-16}$ suffices. These improvements also provide good evidence—but not a proof—that Erdős–Rényi networks with $p \gg \log(n)/n$ globally synchronize with high probability as $n \rightarrow \infty$.

VII. DISCUSSION

We have demonstrated how spectral properties of a graph’s adjacency and graph Laplacian matrix can be used to understand the global synchrony of a Kuramoto model with identical oscillators coupled according to a network. For Erdős–Rényi graphs, we prove that $p \gg \log^2(n)/n$ is sufficient to ensure global synchrony with high probability as $n \rightarrow \infty$. As conjectured by Ling *et al.*,¹⁸ we also believe that the global synchrony threshold is close to the connectivity threshold of $p \sim \log(n)/n$. With the aid of a computer and Lemma 7, we have convincing evidence that Erdős–Rényi networks with $p \gg \log(n)/n$ are globally synchronizing with high probability as $n \rightarrow \infty$, and it is a future challenge to write down a formal proof. While Sec. VI focuses on Erdős–Rényi graphs, most of our analysis applies to any network, and we hope that it can deliver intriguing results for other random graph models.

ACKNOWLEDGMENTS

This research was supported by the Simons Foundation grant (No. 713557) (M.K.), the NSF grants (Nos. DMS-1513179 and CCF-1522054) (S.H.S.), and the NSF grants (Nos. DMS-1818757, DMS-1952757, and DMS-2045646) (A.T.). We thank Lionel Levine and

Mikael de la Salle on MathOverflow for references on bounding $\|\Delta_A\|$ and $\|\Delta_L\|$ for Erdős–Rényi graphs.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Martin Kassabov: Conceptualization (equal); Formal analysis (equal); Methodology (equal); Software (equal); Writing – original draft (equal); Writing – review & editing (equal). **Steven H. Strogatz:** Conceptualization (equal); Formal analysis (equal); Methodology (equal); Software (equal); Writing – original draft (equal); Writing – review & editing (equal). **Alex Townsend:** Conceptualization (equal); Formal analysis (equal); Methodology (equal); Software (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

REFERENCES

- A. T. Winfree, “Biological rhythms and the behavior of populations of coupled oscillators,” *J. Theor. Biol.* **16**, 15 (1967).
- A. T. Winfree, *The Geometry of Biological Time* (Springer, 1980).
- Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, 1984).
- R. E. Mirollo and S. H. Strogatz, “Synchronization of pulse-coupled biological oscillators,” *SIAM J. Appl. Math.* **50**, 1645 (1990).
- S. Watanabe and S. H. Strogatz, “Constants of motion for superconducting Josephson arrays,” *Physica D* **74**, 197 (1994).
- A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, 2003).
- S. Strogatz, *Sync: The Emerging Science of Spontaneous Order* (Hyperion, 2003).
- F. Dörfler and F. Bullo, “Synchronization in complex networks of phase oscillators: A survey,” *Automatica* **50**, 1539 (2014).
- A. Pikovsky and M. Rosenblum, “Dynamics of globally coupled oscillators: Progress and perspectives,” *Chaos* **25**, 097616 (2015).
- F. A. Rodrigues, T. K. D. M. Peron, P. Ji, and J. Kurths, “The Kuramoto model in complex networks,” *Phys. Rep.* **610**, 1 (2016).
- D. M. Abrams, L. M. Pecora, and A. E. Motter, “Introduction to focus issue: Patterns of network synchronization,” *Chaos* **26**, 094601 (2016).
- S. Boccaletti, A. N. Pisarchik, C. I. Del Genio, and A. Amann, *Synchronization: From Coupled Systems to Complex Networks* (Cambridge University Press, 2018).
- M. H. Matheny, J. Emenheiser, W. Fon, A. Chapman, A. Salova, M. Rohden, J. Li, M. H. de Bady, M. Pósfai, L. Duenas-Osorio, M. Mesbahi, J. P. Crutchfield, M. C. Cross, R. M. D’Souza, and M. L. Roukes, “Exotic states in a simple network of nanoelectromechanical oscillators,” *Science* **363**, eaav7932 (2019).
- G. Csaba and W. Porod, “Coupled oscillators for computing: A review and perspective,” *Appl. Phys. Rev.* **7**, 011302 (2020).
- C. W. Wu, “Graph coloring via synchronization of coupled oscillators,” *IEEE Trans. Circuits Syst. I: Fundam. Theory Appl.* **45**, 974 (1998).
- A. Novikov and E. Benderskaya, “Oscillatory network based on Kuramoto model for image segmentation,” in *Proceedings of the 13th International Conference on Parallel Computing Technologies* (Springer, 2015), Vol. 9251, p. 210.
- S. Steinerberger, “Max-cut via Kuramoto-type oscillators,” [arXiv:2102.04931](https://arxiv.org/abs/2102.04931) (2021).
- S. Ling, R. Xu, and A. S. Bandeira, “On the landscape of synchronization networks: A perspective from nonconvex optimization,” *SIAM J. Optim.* **29**, 1879 (2019).
- R. Taylor, “There is no non-zero stable fixed point for dense networks in the homogeneous Kuramoto model,” *J. Phys. A: Math. Theor.* **45**, 055102 (2012).
- J. Lu and S. Steinerberger, “Synchronization of Kuramoto oscillators in dense networks,” *Nonlinearity* **33**, 5905 (2020).
- M. Kassabov, S. H. Strogatz, and A. Townsend, “Sufficiently dense Kuramoto networks are globally synchronizing,” *Chaos* **31**, 073135 (2021).
- P. Erdős and A. Rényi, “On the evolution of random graphs,” *Publ. Math. Inst. Hung. Acad. Sci.* **5**, 1 (1960).
- A. Jadbabaie, N. Motee, and M. Barahona, “On the stability of the Kuramoto model of coupled nonlinear oscillators,” in *Proceedings of 2004 American Control Conference* (IEEE, 2004), Vol. 5, p. 4296.
- D. A. Wiley, S. H. Strogatz, and M. Girvan, “The size of the sync basin,” *Chaos* **16**, 015103 (2006).
- Z. Füredi and J. Komlós, “The eigenvalues of random symmetric matrices,” *Combinatorica* **1**, 233 (1981).
- V. H. Vu, “Spectral norm of random matrices,” *Combinatorica* **27**, 721 (2007).
- M. Barahona and L. M. Pecora, “Synchronization in small-world systems,” *Phys. Rev. Lett.* **89**, 054101 (2002).
- G. S. Medvedev, “Small-world networks of Kuramoto oscillators,” *Physica D* **266**, 13 (2014).
- G. S. Medvedev and X. Tang, “Stability of twisted states in the Kuramoto model on Cayley and random graphs,” *J. Nonlinear Sci.* **25**, 1169 (2015).
- G. S. Medvedev and X. Tang, “The Kuramoto model on power law graphs: Synchronization and contrast states,” *J. Nonlinear Sci.* **30**, 2405 (2020).
- H. Chiba, G. S. Medvedev, and M. S. Mizuhara, “Bifurcations in the Kuramoto model on graphs,” *Chaos* **28**, 073109 (2018).
- H. Daido, “Order function and macroscopic mutual entrainment in uniformly coupled limit-cycle oscillators,” *Prog. Theor. Phys.* **88**, 1213 (1992).
- J. C. Perez and F. Ritort, “A moment-based approach to the dynamical solution of the Kuramoto model,” *J. Phys. A: Math. Gen.* **30**, 8095 (1997).
- Y. Terada and Y. Y. Yamaguchi, “Linear response theory for coupled phase oscillators with general coupling functions,” *J. Phys. A: Math. Gen.* **53**, 044001 (2020).
- G. H. Golub and C. F. Van Loan, *Matrix Computations* (John Hopkins University Press, 2013).
- J. A. Tropp, “An introduction to matrix concentration inequalities,” [arXiv:1501.01571](https://arxiv.org/abs/1501.01571) (2015).
- U. Feige and E. Ofek, “Spectral techniques applied to sparse random graphs,” *Random Struct. Algorithms* **27**, 251 (2005).
- J. Lei and A. Rinaldo, “Consistency of spectral clustering in stochastic block models,” *Ann. Stat.* **43**, 1 (2015).