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David Richeson

BOOK REVIEW

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Single variable calculus as we know it—the ε-δ definition of the limit, the limit definition of continuity and the derivative, the mean value theorem, Riemann's definition of the integral, the fundamental theorem of calculus, power series representations of functions, and so on—is now well over a century old. Calculus has moved from the research frontier to a course taught to high school and college students. Calculus textbooks have settled into an equilibrium. They are all identical; the chapters and sections in competing books often match up nearly perfectly. The only difference from book to book or edition to edition are slight variations in the homework problems, colorful diagrams, and online resources. Is there really anything else to say about this subject?

Three authors think so. Cornell University professor and multi-award-winning author and communicator Steven Strogatz wrote *Infinite Powers*, Macalester College professor and past president of the Mathematical Association of America David Bressoud wrote *Calculus Reordered*, and Ben Orlin followed up his popular debut *Math with Bad Drawings* with *Change Is the Only Constant*.

Scores of students purchase a 1200-page calculus textbook every September. This tome looks impressive and frightening to them. They know it will take an intense year of study to master the techniques of single variable calculus. Little do they know that this book represents the culmination of thousands years of work by history's greatest minds. Giants standing on the shoulders of giants. It could not have been written without earlier work in Euclidean geometry, algebra, analytic geometry, the construction of the real numbers (integers, rational, irrational, and negative numbers), the introduction of transcendental functions, and an understanding of infinity. It is not hyperbole to call calculus one of humankind's greatest achievements.

In a sense, it is amazing that calculus is so *easy* for these students to learn and use. Orlin writes that "By design, calculus is automated thinking." Bressoud writes that "for many students it amounts to little more than mastering manipulations of algebraic expressions. Although such facility is meaningless by itself, the reason that calculus is regarded as *the* preeminent tool for calculation is that such easily memorized procedures can be used to obtain solutions to deep and challenging problems." Orlin reminds us of V. I. Arnold's cynical comment that Leibniz developed calculus "in a form specifically suitable to teach . . . by people who do not understand it to people who will never understand it."

While a large part of the success of calculus is that it has been fine-tuned to the point that it is as mechanical as arithmetic or algebra, it would be a sad state if Arnold were correct. But the ideas behind calculus are subtle and deep, and one year of calculus is not enough to achieve a full understanding of the subject. Moreover, because of the

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way calculus is taught, with an emphasis on techniques, students—even those who excel in the course—often cannot see the big picture.

In their new books, Bressoud, Strogatz, and Orlin succeed in using history, applications, analogies, and plain language to bring this big picture to their readers.

Applications of calculus

In 1988, Underwood Dudley reviewed a 950-page calculus textbook [1]. Instead of discussing the details of this particular textbook, he employed his biting wit to critique the entire genre. He came to seven conclusions such as "Calculus is hard" and "Calculus books are too long." He also criticized the so-called applications presented in these textbooks: "In the 85 calculus books I examined, almost all of them had the Norman window problem—the rectangle surmounted by a semicircle, fixed perimeter, maximize the area. The semicircle always 'surmounts.' This is the sole surviving use of 'surmounted' in the English language except for the silo, a cylinder surmounted by a hemisphere." He concluded that "First-semester calculus has *no* applications."

Strogatz would strongly disagree with Dudley's conclusion. In *Infinite Powers* he gives "an applied mathematician's take on the story and significance of calculus." He refers to "the unreasonable effectiveness of mathematics," a phrase that comes from Eugene Wigner. And he opens his book with a conversation between author Herman Wouk and physicist Richard Feynman that ends with Feynman saying, "You had better learn [calculus]. It's the language God talks." To Strogatz, calculus is everywhere.

To be fair to Dudley, Strogatz's applications require more advanced calculus than what a student would see in one semester, and his applications of calculus are, for the most part, not the type that will end up in the problem section of a textbook. Typically when someone asks, "What is an application of (blank)?" they expect an example in which the connection between the mathematics and the application is clear and apparent—the relationship between the encryption of credit card information on the web and number theory or between Google's page rank system and linear algebra come to mind. Strogatz writes about applications such as kinematics, planetary motion, the longitude problem, general relativity, computerized tomography (CT) scanners, global positioning systems (GPS), HIV modeling, the design of the Boeing 787 Dreamliner, and computer animation. Some are clearly connected to calculus. For others, the calculus is buried deeper down, but it is true that these applications could not exist if it were not for calculus. It is more like telling the story of Jocelyn Bell Burnell's discovery of pulsars in a book about geometry because of the role paraboloids play in radio telescope design. A cynic might view the applications as contrived and a stretch, but a more generous approach is that they show just how fundamental and engrained calculus is in science and technology.

One interesting application that Strogatz mentions that could be used in a calculus class is the discussion of the Olympian Usain Bolt. Strogatz gives us data about Bolt's position at discrete time values for his 2008 record-breaking 100-meter dash. We can ask about Bolt's velocity at various times—for instance, did he really slow down at the end when he knew the gold medal was his? (He did.) Later, Strogatz discusses the more difficult problem of finding Bolt's position at a given time if we were given velocity data. The local problem (differentiation) is easier than the global problem (integration). He also uses Bolt to discuss mathematical models; at the 2009 World Championships researchers used a laser gun to measure Bolt's speed hundreds of times per second. We see that with each step his speed goes up and down, yielding a wiggly velocity curve, which doesn't resemble the smooth curve we would use to model it.

In the 27 short chapters of *Change Is the Only Constant*, Orlin also presents applications of calculus to the real world, although they might more accurately be called stories or vignettes—all illustrated in his now-recognizable "bad drawing" style: stick figures with circular heads and big round eyes. Here is how Orlin explained his book to his English teacher friend: "It's a tour of calculus, . . . but with no fancy equations. No intricate computations. Just the ideas, the concepts—all illustrated by stories. The stories will cut across human experience, from science to poetry, from philosophy to fantasy, from high art to everyday life."

Some of Orlin's stories are familiar topics that are in every calculus course—the product rule, Gabriel's horn, and the fundamental theorem of calculus—although none are presented in the familiar textbook way. Other topics are less likely to be in a calculus course and may be new to calculus teachers: the Laffer curve; Elvis, Timothy Pennings's dog who "knows calculus"; Mary Tai, the medical researcher who rediscovered the trapezoid rule; a footnote in David Foster Wallace's *Infinite Jest* about the mean value theorem and the mean value theorem for integrals. And there other topics that students would not encounter until graduate school, such as the Lebesgue integral and Brownian motion. It is certainly the case that everyone—teachers, students, and interested readers—will learn new mathematics or see familiar ideas in a new way.

Orlin cleverly employs analogies to help the reader understand abstract concepts. In one chapter he uses Leo Tolstoy's *War and Peace* to discuss the idea that integration is the sum of infinitely many infinitesimal values and that the value of a function at a single point does not sway the value of the integral. Tolstoy wrote, "To study the laws of history we must completely change the subject of our observation, must leave aside kings, ministers, and generals, and study the common, infinitesimally small elements by which the masses are moved." That is, individuals do not matter in history; history is the sum total of all of them. Orlin writes, "How do you get from the infinitely small to the unimaginably large? From tiny acts of free will to the unstoppable motions of history? . . . An integral." And later, "The integral is the bridge between Tolstoy's gift and his dream. It's supposed to reconcile the world he knows (a jumble of details) and the world he craves (a well-governed realm), to fuse infinite multiplicity into perfect oneness. Tolstoy's integral fails as science, but I think it succeeds as metaphor."

The mention of Tolstoy is not a one-off detour outside the world of mathematics. Orlin is well read and brings into his writing ideas from all areas of human thought. He name-drops, quotes, or writes about the work of such diverse figures such as Isaac Newton; Zeno of Elea; the physicist Max Planck; an unnamed Chinese philosopher; authors William Faulkner, Jorge Luis Borges, Virginia Woolf, Thomas Wolfe, and Dr. Suess; and these all appeared in the nine pages of chapter 1!

Unlike Strogatz and Orlin, Bressoud does not discuss many applications of calculus. Because *Calculus Reordered* focuses on the history and theory of calculus, he confines his applications to historically important ones, such as kinematics, planetary motion, electricity and magnetism, Fourier series, and the vibrating string.

The history of calculus

The history of calculus is long, rich, and complicated. Some high points mentioned by these authors are the work of Archimedes on the circumference and area of the circle, the volume and surface area of the sphere, and the area of a sector of a parabola; Cavalieri's indivisibles; the development of algebra and analytic geometry; the work of Galileo, Kepler, Huygens, and Newton on problems in physics, astronomy, and mechanics; Napier's logarithm; the early discovery of calculus techniques, such as Fermat's and Descartes's methods for finding extreme values of functions; Newton's and

Leibniz's work bringing these disparate ideas into one cohesive subject; Euler's success in leveraging the power of calculus; Fourier's work with series of sine functions; and Cauchy, Weierstrass, Cantor, Riemann, and Lebesgue's work formalizing calculus.

Not all three authors discuss all of these episodes. And it is interesting what bits of the story the authors leave out and what they include. For instance, the Newton–Leibniz controversy does not appear at all in Bressoud's work. (He mostly writes about Newton and Leibniz separately, and when he mentions them together he writes, "The genius of Newton and Leibniz, and the reason that they are credited as the founders of calculus, is that they were the first to understand and appreciate the full power of understanding integration and differentiation as inverse processes.") Orlin writes a few sentences about it, and Strogatz devotes a few pages to it. (In fact, more memorable to me than the Newton–Leibniz rivalry was Strogatz's discussion of the Fermat–Descartes rivalry.) And although *Infinite Powers* contains a lot about the history of calculus, Strogatz ends the usual historical progression at Newton and Leibniz; Euler and Cauchy do not appear once, and Weierstrass and Riemann are mentioned only in passing.

Calculus Reordered is full of historical gems. For instance, Bressoud writes about the Indian astronomer Aryabhata's method, which dates to the end of the fifth century CE, of computing the sine function. By using trigonometric identities, Aryabhata could compute $\sin\theta$ for multiples of 3°. To find the value at other angles, he could use interpolation; this would essentially be using a secant line to the graph of the sine function. Instead, he used linear approximation. Aryabhata showed that as long as we use the same units to measure the arclength and the half-chord (or, as we tell our students, as long as we use radians and not degrees), then we get the improved approximation $\sin(\theta + \Delta\theta) \approx \sin(\theta) + \cos(\theta)\Delta\theta$. Bressoud concludes, "One could claim that the first function to be differentiated was the sine, it happened in India, and it occurred well over a thousand years before Newton and Leibniz were born."

Teaching calculus

Today's textbooks do an admirable job of teaching the mechanics of calculus. At the end of the year, students can apply the quotient rule, integrate by parts, and use l'Hôpital's rule. But do they truly understand what calculus *is*? It is not as clear that textbooks get these ideas across. And what about people who have never studied calculus and those who took calculus many years ago? Surely we would not recommend that they read a calculus textbook if they wanted to know about calculus.

Anyone who is familiar with Steven Strogatz knows that he is a knowledgeable mathematician, a story teller who humanizes mathematics, a mathematician who cares about the history of the subject, and a patient and thoughtful teacher. Thus, it is a treat that he chose to write about calculus for his most recent book. His teaching skills are in full force, and readers will come away with a much better understanding of calculus after reading *Infinite Powers*. And, although his book is not appropriate for use in a calculus class, any teacher who reads it will gain a better perspective of the field and will come away with ideas and talking points to use in the classroom.

As a former high school teacher, Orlin intended to have *Change Is the Only Constant* follow the AP calculus curriculum. But, he writes, "The more I followed my map, the queasier I became. I wanted a yellow brick road of math, bursting with color and magic; this felt more like the trail through an Ikea." Thus, he changed course and wrote a story-based book that does not follow the curriculum. But, with teachers in mind, he added an appendix that maps his chapters to the standard calculus curriculum.

Unlike Strogatz and Orlin, Bressoud is writing for calculus teachers. He believes that teachers must know the history and theory behind the subject in order to teach it effectively. Moreover, as the title *Calculus Reordered* implies, he believes that there is a better progression through the material than the standard one.

Virtually every single variable calculus textbook proceeds in the same order: limits, differentiation, integration, and infinite series. This seems like the right order. We need limits in order to introduce any of the other ideas. So it must come first. Derivatives are easier to compute than integrals, so they are taught next. After introducing Riemann sums and defining the integral, we present the fundamental theorem of calculus. This theorem turns the problem of integration into the problem of finding antiderivatives, which is more difficult than differentiation but easier than computing limits of Riemann sums. We save infinite series and power series for last.

But this is not the order that calculus was discovered. Writing about the derivative in particular, the historian of mathematics Judith Grabiner famously wrote [2] that it "was first *used*; it was then *discovered*; it was then *explored and developed*; and it was finally *defined*." And moreover, the historical order of discovery of the four big ideas of calculus—first integration, then differentiation, then infinite series, and then limits—is almost the exact opposite of the order they are taught. Bressoud thinks that they should be taught in the order they were discovered.

He argues that "The progression we now use is appropriate for the student who wants to verify that calculus is logically sound. However, that describes very few students in first-year calculus." Moreover, our calculus courses are not rigorous. They are like the Hollywood movie sets in which the buildings are wooden facades. The truly rigorous approach has to wait for an upper-level course in real analysis.

Bressoud believes that we should be begin with integration, or accumulation, as he prefers to call it. It is the oldest idea of calculus, it arose from geometry, and there are many interesting, intuitive, and useful examples. He contends that our current curriculum moves too quickly from integration to antiderivatives: "students who think of integration as primarily reversing differentiation often have trouble making the connection to problems of accumulation." And because of the power of technology, techniques of integration are skills that are increasingly used only in a calculus class.

For the derivative, Bressoud suggests focusing more on ratios of change than on the slope of a tangent line. "It is common to introduce the derivative as the slope of a tangent line. Such an approach can create pedagogical difficulties. Philosophers and astronomers worked with ratios of change long before anyone computed slopes. Students who do not understand slope as a ratio of changes can fail to appreciate the full power of this concept."

For series, Bressoud suggests we focus on partial sums, and Taylor polynomials in particular. He prefers teaching the Lagrange remainder theorem rather than issues of convergence, which can be pushed off to a real analysis course.

Limits are problematic in our calculus courses. The ε-δ definition is not appropriate in first-year calculus, but our current method of teaching limits focusses on an x-first approach whereas the definition is y-first. Of course, limits were also a challenge historically. Archimedes and the Greeks gave double proofs by contradiction, showing that a value is neither more more less than the desired quantity. Each such argument was tailor-made for the problem at hand. From Kepler through Leibniz, the Bernoullis, and Euler, limiting arguments were made using infinitesimals. There was a lot of unease about such arguments, although they gave the correct answers. Some, like Newton and d'Alembert, used language such as "tends to" or "approaching." The now-familiar ε-δ approach, what Bressoud calls the algebra of inequalities, had to wait for Cauchy in the 19th century. Thus, Bressoud suggests ending the calculus sequence with limits.

Target audiences

I have two friends who are not mathematicians—a financial advisor and a lawyer. Both of them use logic and quantitative skills in their jobs, but although they took mathematics courses in college, they no longer do "school math." They are voracious readers and consumers of nonfiction, and mathematics is one of their areas of interest. They are always telling me about the latest book they read and asking me to elucidate some mathematical nugget they heard about. Strogatz's and Orlin's books were written for them. As Strogatz writes, "It isn't necessary to learn how to do calculus to appreciate it, just as it isn't necessary to learn how to prepare fine cuisine to enjoy eating it."

Both authors write in a very conversational way, occasionally in first person, and pop culture references are sprinkled liberally throughout the text. For example, Strogatz writes, "many books keep infinitesimals locked in the attic, like Norman Bates's mother in *Psycho*. But they're really nothing to be afraid of. Really. Let's go meet Mother." (Strogatz did add on Twitter "BTW, a friend of mine corrected one small detail: Mother was kept in the cellar, not the attic. That will be fixed in the next printing.") Orlin, writing about Laplace's scientific determinism, channels Shakespeare: "All the world's a differential equation, and the men and women are merely variables." In both books, the low-key conversational approach is extremely effective. Reading these books is like talking to a friend. They do not talk down to the reader nor do they talk over their heads; they truly want the readers to understand what calculus is.

Much like good a filmmaker will throw adult humor into a kids' movie, Strogatz and Orlin include ideas that would interest mathematicians while writing for readers who do not know much mathematics. Nonmathematicians will get a feeling for what calculus is and how it is used, and mathematicians will gain deeper insights into the material they teach every day.

I would not recommend Bressoud's book to my two friends, but I would recommend it to my colleagues who teach calculus. It is not overly dense or technical, but it also does not waste time putting things in context—context that a mathematician would not need. Reading *Calculus Reordered* is also like talking to a friend, but in this case it is like talking to a friend who is also a math teacher. Bressoud's book would be a valuable book for any calculus teacher to read. The historical background of the mathematics is fascinating, and it opens the door to the larger question of how, and in what order, we should teach calculus.

None of these books will help a student pass calculus (except to get them more interested in the subject or to give them the idea of "why?"). Yet, everyone who teaches calculus would get something out of all three books—if not just ways to think about and talk about calculus. After reading these wonderful books by Bressoud, Strogatz, and Orlin, we conclude that there *is* more to say about calculus than what is found in textbooks, and we are fortunate to have three such clear expositors to share their expertise in this important subject.

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